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Generalized Supplemented Semimodules

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Abstract: ***In this work generalized supplemented semimodules are defined which generalize generalized supplemented modules. We investigate some properties of these semimodules. We show that the finite sum of GS-semimodules is GS-semimodule. We also define WGS-semimodules and proved that a semiring S is semilocal if and only if every finitely generated semimodule is a WGS-semimodule. Furthermore, we prove that if A be a semimodule as well as $Rad(A) \ll A$. Then A is a WGS-semimodule if and only if $A/Rad(A)$ is semisimple.***

Keywords: ***Generalized Supplemented Semimodule, Subtractive Semimodule, Lifting Semimodule.***

1. INTRODUCTION

In 2006, Wang and Dang [11] defined generalized supplemented module. In this paper, we first introduce the concept of generalized supplemented semimodules. Firstly, let use point that, S shalt indicate an associtione semiring with neutrai and A shalt indicate an unitary left S-semimodule throughout this article. A (left) S-semimodule A is commutative additive semigroup with zero elements 0_A, jointly with a mapping from $S \times A$ into A (sending (s, a) to sa) such that $(r + s)a = ra + rs$, $r(a + b) = ra + rb$, $r(sa) = (rs)a$ and $0_a = s0_a = 0$ for all $a, b \in A$ and $r, s \in S$. Let N be sub set of A. We say that N is an S-subsemimodule of A, denoted by $N \leq A$, if and only if N is itself an S-semimodule with respect to the process for A [6]. A subsemimodule $N \leq A$ is called essential in A, denoted by $N \leq_e A$ (or $N \leq_e A$), if $N \cap L \neq 0$ for every non-zero subsemimodule $L \leq A$ [9]. A subsemimodule $N \leq A$ is said to be small or superfluous in A (writes $N \ll A$), if for every subsemimodule $K \leq A$ with $N + K = A$ assume that $K = A$ [10]. The radical of S-semimodule A, signified by Rad(A), is the sum of all small subsemimodules of A [10]. A is said to be hollow, if all proper subsemimodules of A are small in A. A is called local if it contains the maximal single subsemimodule, the correct subsemimodule that contains all other subsemimodules. A is called simple if it has no nontrivial

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subsemimodules, and semisimple if A is a direct sum of its simple subsemimodules [8]. A semiring S is named semilocal semiring if $S/Rad(S)$ is semisimple. The socle of A meaning $Soc(A)$, is the sum of all simple subsemimodules of A [8]. Let $U,K \leq A$. K is called a supplement of U in A if it is minimal with respect to $A = U + K$. A subsemimodule K of A is a supplement from U in A if and only if $A = U + K$ and $U \cap K \ll K$ [3]. A is supplemented if each subsemimodule U of A has a supplement in A. $U \leq A$ has ample supplements in A if each subsemimodules of A such that $A = U + K$ contains a supplement of U in A. A semimodule A is called amply supplemented if every subsemimodule from A has ample supplements in A. Hollow semimodules are amply supplemented [3]. $U \leq A$ is a subtractive subsemimodule of A if $a, a + b \in U$ then $b \in U$ [8]. If every $U \leq A$ is a subtractive, then A is named subtractive. If C is a subtractive subsemimodule, then A/C is an S-semimodule [6, p.165]. Section 2 is devoted to various properties of generalized supplement subsemimodules. We prove that if and only if A is a GAS-semimodule, then A is Artinian and confirms that DCC generalized to smaller subsemimodules. It is proved so as to every finite sum of GS-semimodule. In secation 3, we define WGS- semimodule. It is proved that S is semilocal iff each cyclic semimodule is a WGS-semimodule.

Lemma 1.1: Let A be a semimodule and V a supplement subsemimodule of A. Then $Rad(V) = V \cap Rad(A)$.

Proof: Assume V to be a supplement of $U \leq A$. Let $K \ll A$ and $Y \leq V$ with $(K \cap V) + Y = V$. The Then $A = U + V = U + (K \cap V) + Y = U + Y$, and so $Y = V$, i.e., $K \cap V \ll V$. This yields $V \cap Rad(A) \leq Rad(V)$, since $Rad(V) \leq V \cap Rad(A)$, we have $Rad(V) = V \cap Rad(A)$. □

In [6], [8] if A is a semimodule, then A represent Artinian if any non-empty set on subsemimodules of A contains minimal member in ration to setting inclusion. For present definition is equivalent to descending chain condition on subsemimodules of A.

Theorem 1.2: Take A a S-semimodule. If and only if A fulfills (DCC) on a small submodule, $Rad(A)$ is then Artinian.
Proof: It is essentially the same as that of Theorem 5 in [2]. □

Proposition 1.3: [11, Proposition 14.22] (Semimodularity Law) Let A be an S-semimodule and let N_t and N_r be subsemimodules of A. Let F be a subtractive subsemimodule of A with $N_t \leq F$. Then $F \cap (N_t + N_r) = N_t + (F \cap N_r)$.

2. GS-semimodules and GAS-Semimodules

Definition 2.1: Let A be an S-semimodule and $F, B \leq A$. If $A = F + B$ and $F \cap B \leq Rad(B)$, then B is said to be a generalized supplement of F in A. If all subsemimodule of A has a generalized supplement in A, then A is named generalized supplemented semimodule or simply a GS-semimodule.

Definition 2.2: An S-semimodule A is said to be a generalized amply supplemented semimodule or simply a GAS-semimodule when $A = F + B$ (mean that F has generalized

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supplement $F' \leq B$). F is named a generalized supplement subsemimodule when F is a generalized supplement of some subsemimodule of A.

Remark 2.3: (1) Evidentially, all supplement is generalized supplement. Therefore, all supplemented semimodules are generalized supplemented. But the reverse include is incorrect. For instance, the Z-semimodule Q. Since $\mathrm{Rad}(Q) = Q$. While Q is not supplemented using example [5, 20.12].

(2) (Amply) supplemented semimodules and hollow semimodules are GS-semimodules. Let $S = N$ be a semiring of non-negative integers, and $A = N_8 = \frac{N}{8N}$. Then the semimodules S and A over a semiring S are local (hollow) semimodules and so are GS-semimodules.
The next can be thought of as a generalization of [4, Example 7.5]

Example 2.4: Let S be a Dedekind semidomain of the quotient semifield $K \neq S$. The S-semimodule $A = K^{(\Gamma)}$is generalized supplemented for all index Γ. If S is (a local Dedekind) semidomain, then A is supplemented only when Γ is finite. If S is a non-local Dedekind semidomain, as well A not supplemented for each index set Γ, because A not torsion, i.e., $\mathrm{Rad}(A) \neq A$.

Proposition 2.5: Take A be a GS-semimodule and N a subtractive subsemimodule of A such that $N \cap \mathrm{Rad}(A) = 0$. As well N is semisimple. Exclusively, a GS-semimodule A to $\mathrm{Rad}(A) = 0$ is semisimple.
Proof: Let $N^1 \leq N$. There exists $N^2 \leq A$ with $N^1 + N^2 = M$, $N^1 \cap N^2 \leq \mathrm{Rad}(N^2)$. So $N = N \cap A = N \cap (N^1 + N^2) = N^1 + (N \cap N^2)$, by Proposition 1.3. Since $N^1 \cap N^2 \leq \mathrm{Rad}\,(N^2)$ and $N^1 \cap N \cap N^2 = N^1 \cap N^2 \leq N \cap \mathrm{Rad}\,(N^2) \leq N \cap \mathrm{Rad}(A) = 0$, $N = N^1 \oplus (N \cap N^2)$. So that N would be is semisimple. □

Proposition 2.6: Take A subtractive GAS-semimodule also take Fdirect summand of A. As well F would be GAS-semimodule.
Proof: There exists $F' \leq A$ with $A = F \oplus F'$. Assume $F = B + H$, then $A = H + (B \oplus F')$. Since A is a GAS-semimodule, there exists $K \leq H$with $A = K + (C \oplus F')$ and $K \cap (B \oplus F') \leq \mathrm{Rad}(K)$. So $F = F \cap A = F \cap \big(K + (B \oplus F')\big) = K + B$, $K \cap B = K \cap (B \oplus F') \leq \mathrm{Rad}(K)$, This proves the claime. □

Proposition 2.7: Let A be a subtractive GS-semimodule. Then $A = F \oplus B$ to some semisimple F also some B with essential radical.
Proof: It is similar to the proof of Proposition 2.3 in [11]. □

Proposition 2.8: Let A_1, $U \leq A$ and A_1 be a GS-semimodule. If $A_1 + U$ contains a generalized supplement in A, furthermore U.
Proof: There exists $N_t \leq A$ with $N_t + (A_1 + U) = A$, $N_t \cap (A_1 + U) \leq \mathrm{Rad}\,(N_t)$. There exists $N_k \leq A_1$ with $(N_t + U) \cap A_1 + N_k = A_1$, $(N_t + U) \cap N_k \leq \mathrm{Rad}(N_k)$. So we have $N_t + U + N_r = A$ and $(N_t + U) \cap N_r \leq \mathrm{Rad}(N_r)$. It is sure that $(N_t + N_r) + U = A$. Since $N_r + U \leq$

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$A_1 + U, N_t \cap (N_r + U) \leq N_t \cap (A_1 + U) \leq Rad(N_t)$. Hence $(N_t + N_r) \cap U \leq N_t \cap (N_r + U) + N_r \cap (N_t + U) \leq Rad(N_t) + Rad(N_r) \leq Rad(N_t + N_r)$. So, $N_t + N_r$ be generalized supplement of U in A. □

Proposition 2.9: Let $A_1$and A_2 be GS-semimodules. If $A = A_1 + A_2$, then A is a GS-semimodule.
Proof: Let $U \leq A$. Since $A = A_1 + A_2 + U$ trivially contains a generalized supplement in A, $A_2 + U$ contains a generalized supplement in A using Proposition 2.8. Then also U using Proposition 2.8. □

Theorem 2.10: If A is a GS-semimodule, then $A/Rad(A)$ is semisimple.
Proof: Let $N_t \leq A$ with $Rad(A) \leq N_t$. Then $A = N_t + N_k$ and $N_t \cap N_k \leq Rad\ (A)$ for some $N_k \leq A$. So $A/Rad(A) = N_t/Rad(A) \oplus (N_k + Rad(A))/Rad(A)$, as well as all subsemimodule of $A/Rad(A)$ is a direct summand. □

Definition 2.11: A subsemimodule F of A is said to have generalized ample supplements in A if for all $H \leq A$ with $F + H = A$, F has a generalized supplement in H.

Proposition 2.12: If $A = A_1 + A_2$, and A_1, A_2 have generalized ample supplements in A, then $A_1 \cap A_2$ also has generalized ample supplements in A.

Theorem 2.13: The next are equivalent for a subtractive semimodule A with $B \leq A$.
(1) There is a decomposition $A = F \oplus F'$ with $F \leq B$ and $F' \cap B \leq Rad(F')$.
(2) There is a direct summand Fof A with $F \leq B$ and $B/F \leq Rad(A/F)$.
(3) B has a generalized supplement H in A with $H \cap B$ is a direct summand of B.

Proof: (1) $\Longrightarrow$ (2) Using the subtractiveness of F, we have A/F is a semimodule. $B/F \cong F' \cap B \leq Rad(F') \cong Rad(A/F)$. So $A = F \oplus F'$ and $B/F \leq Rad(A/F)$.
(2) $\Longrightarrow$ (1) If $A = F \oplus F'$ and $B/F \leq Rad(A/F)$, then $B = F + (F' \cap B)$, $F' \cap B \cong \frac{B}{F} \leq Rad\left(\frac{A}{F}\right) \leq A/F \cong F'$, consequently $F' \cap B \leq Rad(F')$.

(1) $\Longrightarrow$ (3) By hypothesis, F' the generalized supplement of B in A and $B = F \oplus (F' \cap B)$.
(3) $\Longrightarrow$ (1) Take H be a generalized supplement of B. Let $B = F \oplus (H \cap B)$. Then $A = B + H = F + (H \cap B) + H = F +$ Hand $F \cap H = (F \cap B) \cap H = F \cap (H \cap B) = 0$ (for $F \leq B$), i.e. F is a direct summand of A. □

Proposition 2.14: If each subsemimodule of A is a GS-semimodule, then A is a GAS-semimodule.
Proof: Let $F, B \leq A$ as well as $A = B + F$. There is $H \leq F$ with $F \cap B + H = F$, $(F \cap B) \cap H = B \cap H \leq Rad(H)$. So, $H + (F \cap B) = F \leq H + B$, so $B + F = A \leq H + B$. $A = H + B$. □

Corollary 2.15: The next are equivalent, for a semiring S.
(1) Each semimodule is a GAS-semimodule.

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(2) Each semimodule is a GS-semimodule.

Definition 2.16: [3] A semimodule A is named π-projective if for any two subsemimodules F and B of A with $A = F + B$, there is $\varphi, \gamma \in End(A)$ such that $\varphi + \gamma = 1_A$, $\varphi(A) \leq F$ and $\gamma(A) \leq B$.

Theorem 2.17: If A is a subtractive π-projective GS-semimodule, we get A is a GAS-semimodule.
Proof: Similar to the proof of theorem 3.16 in [3]. □

Theorem 2.18: A is Artinian if and only if A is a GAS-semimodule as well as fulfills (DCC) on generalized supplement subsemimodules with on small subsemimodules.
Proof: The first trend obviously. Reverse direction, assume A GAS satisfies (DCC). Thus $Rad(A)$ is Artinian by Theorem 1.2. Let $F \leq A$ and $Rad(A) \leq F$. There is a generalized supplement H of F in A, i.e., $A = F + H$, $F \cap H \leq Rad(H) \leq Rad(A)$. So $[\frac{A}{Rad(A)} = (F/Rad(A)) \oplus ((H + Rad(A))/Rad(A))]$. $A/Rad(A)$ is semisimple.

Now assume $Rad(A) \leq F_1 \leq F_2 \leq F_3 \leq \cdots$ represent ascending chain of subsemimodules of A. Since A is a GAS-semimodule, we can find a descending chain of subsemimodules $H_1 \geq H_2 \geq \cdots$ together with H_i is a generalized supplement of F_i in A to all $i \geq 1$. By assumption, there is positive integer r together with $H_r = H_{r+1} = H_{r+2} = \cdots$. Since $A/Rad(A) = F_i/Rad(A) \oplus (H_i + Rad(A))/Rad(A)$ for all $i \geq r$, it follows that $F_r = F_{r+1} = \cdots$. From now, $\frac{A}{Rad(A)}$ is Noetherian, so finitely generated. Therefore $A/Rad(A)$ is Artinian, □

Corollary 2.19: If A is finitely generated GAS-semimodule. Then A is Artinian iff A fulfills (DCC) on small subsemimodules.
Proof: "$\Longrightarrow$" is clear.
"$\Longleftarrow$" As $A/Rad(A)$ is semisimple and A is finitely generated, so $A/Rad(A)$ is Artinian. Since A fuifills (DCC) on small subsemimodules, $Rad(A)$ is Artinian using Theorem 1.2. Therefore, A is Artinian. □

Definition 2.20: [9] A semimodule A is called a lifting semimodule if for every subsemimodule $F \leq A$ of A there exist subsemimodules H, H' for A as well as $A = H \oplus H'$, $H \leq F$ and $F \cap H' \ll H'$.

Definition 2.21: [9] Let A be a subtractive semimodule. Then A is said to be a lifting semimodule, if for every subsemimodule $F \leq A$, there is a direct summand H of A and $H \leq A$ as well as $\frac{F}{H} \ll \frac{A}{H}$.

Theorem 2.22: If A be a subtractive semimodule as well as (ACC) on small subsemimodules. then, A is a GAS-semimodule as well as each generalized supplement is a direct summand of A if and only if A is lifting semimodule.

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Proof: ($\Longrightarrow$) Let $A = F + H$. There is $B \leq H$ with $A = F + B$ and $F \cap B \leq Rad(B)$. Since A together with (ACC) on small subtractive subsemimodules, $Rad(B)$ is Noetherian by [2]. So $Rad(B) \ll B$ by [7, Coro. 9.1.3], as well as B is a supplement of F. So A is amply supplemented. For any supplement is generalized supplement, any supplement is direct summand. A is lifting.

($\Longleftarrow$) Because A is lifting, A is an amply supplemented, from now A is a GAS-semimodule. Assume F be a generalized supplement, i.e., there is $H \leq A$ therefore $A = F + H$ while $F \cap H \leq Rad(F)$. With similar evidence from those of ($\Longrightarrow$), we are aware of that F is a supplement of H. Thus H is a direct summand of A, this proves the claime. □

Remark 2.23: Let A be a GS-semimodule and $Rad(A)$ be Noetherian. Then A is a supplemented semimodule.

3. WGS-semimodules

Definition 3.1: Let F, $H \leq A$. If $A = F + H$and $F \cap H \ll A$, then H is named a weak supplement of F in A. If all subsemimodule of A contains a weak supplement in A, then A is named a weakly supplemented semimodule.

Definition 3.2: A semimodule A is called generalized weakly supplemented or abbreviation writes a WGS-semimodule if for each subsemimodule $F \leq A$, there is $H \leq A$ together with $A = F + H$ and $F \cap H \leq Rad(A)$.

Proposition 3.3: Take A is a WGS-semimodule. Then.
(1) Each supplement subsemimodule of A is a WGS-semimodule.
(2) Each factor semimodule of A is a WGS-semimodule.

Proof: (1) Considered H be a supplement in A. To all $F \leq H$, because A is a WGS-semimodule, there is $B \leq A$ with $A = F + B, F \cap B \leq Rad(A)$. Therefore $H = H \cap A = H \cap (F + B) = F + (H \cap B)$ and $F \cap (H \cap B) = F \cap B = H \cap (F \cap B) \leq H \cap Rad(A) = Rad(H)$ using Lemma 1.1. Hence H is a GWS-semimodule.
(2) Take $B/F \leq A/F$. For $B \leq A$, there is $H \leq A$ together with $B + H = A$ and $H \cap B \leq Rad(A)$ since A is a WGS-semimodule. So, $A/F = B/F + (H + F)/\mathrm{F}$. Let $\varphi: A \to A/F$ be a canonical epic. Since $H \cap B \leq Rad(A)$, $(B/F) \cap ((H + F)/F) = (B \cap (H + F))/F = (F + (H \cap B))/F = \varphi(B \cap H) \leq \varphi(Rad(A)) \leq Rad(A/F)$, A/F is a WGS-semimodule. □

Corollary 3.4: Take A be a semimodule and $F \ll A$. Then A is a WGS-semimodule if and only if $\frac{A}{F}$ is a WGS-semimodule.

Proposition 3.5: Take A is finitely generated. Then A is a WGS-semimodule if, and only if, A is weakly supplemented.
Proof: ($\Longleftarrow$) It's simple.

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($\Longrightarrow$) Suppose $N_t \leq A$, there exists $N_r \leq A$ with $N_t + N_r = A$ and $N_t \cap N_r \leq Rad(A)$ since A is a WGS-semimodule. Since A is finitely generated, $Rad(A) \ll A$ [10]. Hence $N_t \cap N_r \ll A$. □

Lemma 3.6: Let H, $A_1 \leq A$ and A_1 be a WGS-semimodule. If $A_1 + H$ has a generalized weak supplement in A, also H.

Proof: Suppose $F \leq A$ will be $(A_1 + H) + F = A$ and $F \cap (A_1 + H) \leq Rad(A)$. Since A_1 is a WGS-semimodule, there is a $B \leq A_1$ together with $A_1 \cap (F + H) + B = A_1$ and $B \cap (F + H) \leq Rad(A_1)$. Thus $A = H + F + B$ and $H \cap (F + B) \leq (H + A_1) \cap F + B \cap (F + H) \leq Rad(A)$, that is, $F + B$ is a generalized weak supplement of H in A. □

Proposition 3.7: Suppose $A = A_1 + A_2$. If A_1, A_2 are WGS-semimodules, then A is a WGS-semimodule.

Theorem 3.8: Let A be a semimodule as well as $Rad(A) \ll A$.Then the next are equivalent.
(1) A is a WGS-semimodule.
(2) $\frac{A}{Rad(A)}$ is semisimple.
(3) $A = A_1 \oplus A_2$ together with A_1 is semisimple, $Rad(A) \leq_e A_2$ and $A_2/Rad(A)$ is semisimple.

Proof: (1) $\Longrightarrow$ (2) Let $B \leq A$ with $Rad(A) \leq B$. Since A is a WGS-semimodule, there exists $F \leq A$ with $F + B = A$ and $F \cap B \leq Rad(A)$. From now $\frac{A}{Rad(A)} = B/Rad(A) + \frac{(F+Rad(A))}{Rad(A)}$ and $\frac{B}{Rad(A)} \cap \frac{(F+Rad(A))}{Rad(A)} = \big(B \cap F + Rad(A)\big)/Rad(A) = 0$.
(2) $\Longrightarrow$ (1) For any $F \leq A$, since $\frac{A}{Rad(A)}$ is semisimple, there is $B \leq A$ containing $Rad(A)$ as well as $\frac{A}{Rad(A)} = \frac{(F+Rad(A))}{Rad(A)} \bigoplus \frac{B}{Rad(A)}$. Hence $A = F + \mathrm{Rad}(A) + B$. Since $Rad(A) \ll A$, $A = F + B$. $F \cap B \leq Rad(A)$ is clear.
(2) $\Leftrightarrow$ (3) By [11, Theorem 3.8]. □

Theorem 3.9: The next are equivalent, for a semiring S.
(1) S is semilocal.
(2) Each semimodule together with small radical is a WGS-semimodule.
(3) Each finitely generated semimodule is a WGS-semimodule.
(4) Each cyclic semimodule is a WGS-semimodule.

Proof: (1) $\Longrightarrow$ (2) Because every semimodule A there is a set (Γ) as well as an epimorphism $\varphi : S^{(\Gamma)} \longrightarrow A$ together with $\varphi(Rad(S^{(\Gamma)})) \leq Rad(A)$ and $\frac{S^{(\Gamma)}}{Rad(S^{(\Gamma)}) \cong (S/J(S))^{(\Gamma)}}$, an epimorphism exists $\xi : S^{(\Gamma)}/Rad(S^{(\Gamma)}) \longrightarrow \frac{A}{Rad(A)}$. $\frac{A}{Rad(A)}$ is hence semisimple. and therefore A is a WGS-semimodule by applying Theorem 3.8.
(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) the proof obviously.

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(4) $\Longrightarrow$ (1) By Proposition 3.5, since a semiring S is semilocal if and only if ${}_SS$ weakly supplemented. □

Example 3.10: Let p and q be prime numbers and consider the semiring $S = Z_{p,q} = \{\frac{x}{y} \mid x, y \in Z, y \neq 0, p \nmid y \text{ and } q \nmid y\}$, where S is a uniform semilocal Noetherian semidomain. Hence, ${}_SS$ is a WGS-semimodule by using Theorem 3.8. Because ${}_SS$ is Noetherian, it together with (ACC) to small subsemimodules. If ${}_SS$ a GS-semimodule, ${}_SS$ is therefore a supplemented semimodule using Remark 2.23, this results in a note of contradiction in [1, Example 2. 17].

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