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Numerical Simulation of Ordinary Differential Equation by Euler and Runge–Kutta Technique

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***Abstract:** In this paper, we explore Modified Euler's technique & classical Runge Kutta technique of order 4th. These numerical techniques are employed to provide an approximate solution to an initial value problem with ordinary differential equations. These approaches are certainly effective and practically good for solving ordinary differential equations, & they are all used to evaluate degree of accuracy of each approach. We create a table of approximate solution and exact solution comparisons to acquire and assess the level of accuracy of numerical data. The exact and approximate solutions show good agreement, and we compare the computational effort required by the proposed methods. Additionally, we observed that numerical solutions with very short step sizes provide more accurate results. We now identify the errors in the suggested methods and graphically illustrate them to demonstrate their superiority over one another. The Runge-Kutta 4th order approach is more effective in terms of results and also produces less error.*

***Keywords:** Modified Euler's Technique, Classical Runge- Kutta Technique, Error Analysis.*

1. INTRODUCTION

The engineering problems that we don't know came from a lot of different perspectives. There is an important role of numerical methods in solving the problems with a good experience in science and engineering fields. Numerous real-world issues are transformed into differential equations in the area of engineering mathematics. We take small step size h in both the techniques. The original conditions are scaled at the initial points only are labelled initial value problems [5,11]. Numerous scientists investigate the principles of differential equations and other common patterns that cannot be adequately explained using mathematical language. To

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simple differential equations we may obtained close form solution [2]. On the other way many differential equations so much, complicated that can't be expressed in closed form solution. In each scenario, numerical techniques offer additional tools for resolving differential equations given the initial conditions. [2].

Leonhard Euler created the ancestor of all numerical techniques used currently between 1768 and 1770 [12]. Runge-Kutta and modified techniques were first implemented in 1895 and 1905, respectively, by Carl Runge and Martin Kutta [11]. We have excellent and comprehensive publications on topic that can be consulted, like [1,15,24,25,4-10]. In [21] Islam argued that initial value problems to ordinary differential equations may be accurately solve numerically using Runge-Kutta technique of fourth order, whereas [20] provided accuracy analysis of such a solution [13,16,18,22,28]. Without using restrictive transformations or discretization, this study employs Euler's & Runge-Kutta technique of fourth order.

Following a review of the literature, some authors have taken the initiative to maintain high accuracy in their solutions to initial value problems. We are aware of the differential equation in one of the domains of numerical analysis by the names of the widely used algorithms, modified Euler's technique and Runge-Kutta technique. When there is no analytical solution to the problem, one of the key functions of numerical analysis is to provide a numerical response. The analysis of numerical technique problems is a crucial component of learning numerical analysis. So, in order to calculate the effectiveness of a procedure, forecasting errors is necessary. The first numerical technique Euler technique. This method can be understood in the simple way and geometrically smooth to express. In this method we need to take small value h due to this the method is not good for practically. If we take h large this method is not accurate. This approach is most popular since it offers trustworthy starting points, is especially appropriate when computing higher derivatives is challenging [20], & numerical results are highly encouraging.

Finally, examples' findings demonstrate the convergence & error analysis's ability to clearly demonstrate effectiveness of techniques we employed. Euler's method is a single-step, straightforward process that requires not much time. Because $\frac{dy}{dx}$ i.e., f (x, y) varies quickly in the Euler's approach, there is significant difference between computed & exact value. The benefits of Runge-Kutta method include its ability to calculate difficult higher order derivatives with accuracy from the very beginning. Runge-Kutta technique accuracy is higher than modified Euler's technique, & other benefit of this method is that it just needs function value at some points on sub-intervals. Due to its constant starting values and the fact that its results converge more closely to the analytical solution than those of other two techniques, Runge-Kutta method is best-fitting numerical method.
The paper is organized as follows: section 2 present problem formulation; section 3 present numerical examples; section 4 discusses results; & section 5 present study's conclusion.

Problem Formulation

The approximate solution of I.V.P. of differential equation of form is derived in this section with two numerical approaches.

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y'(x) = f(x,y) x ε (a,b) y(a) = y_o (2.1)

where y'(x) = $\frac{dy}{dx}$, f(x,y) is given function & y(x) is solution of (2.1)

Modified Euler Method

This method uses a line whose abscissa is the average of x_0 & x_1 to approximate curve on an interval (x_0, x_1) where $x_1 = x_0 + h$. The generalised Modified Euler technique is $y_{n+1}(x) = y_n(x) + h\, f(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)$)

Runge – Kutta Method

Runge-Kutta methodology is a technique to solve differential equations. By 1894, Karl Runge and Wilhelm Kutta, two mathematicians, developed this method. Because it yields precise and effective results, Runge-Kutta method is well-liked.

The general solution for Runge – Kutta method of 4^{th} order is

$K_1 = hf(x_n, y_n) = hf$

$K_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$

$k_4 = hf(x_n + h, y_n + k_3)$

$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Numerical Examples

In this part of the paper, we take two examples and solve them by using the proposed methods and we also determine that which method converges fast to the actual solution. Then we find error for proposed techniques & represents results, error graphically. Mathematically convergence of numerical method is defined as $\lim_{h\to 0} \max_{1\le n\le N} |y(x_n) - y_n| = 0$ & error is defined as $\lim_{h\to 0} \max_{1\le n\le N} |y(x_n) - y_n|$ where $y(x_n)$ is approximate solution & y_n is actual solution.

Example (1)

Let's assume differential equation $\frac{dy}{dt} = 5t^2 + 2t - 5y$, $y(0) = \frac{1}{3}$ in interval $0 \le t \le 1$ Actual solution of above differential equation is $y(t) = t^2 + \frac{1}{3e^{5t}}$

Now, we obtained an approximate solution of above differential equation by using proposed methods by taking h = 0.1 & find error estimation of proposed technique and finally draw graphs of answers of above differential equation as shown below.

Table 1: Approximate solution of proposed two techniques

N	T_n	Modified Euler's Technique	Runge – Kutta Technique of Order 4^{th}	Exact Solution
0	0.00	$333333333333\times 10^{-12}$	$333333333333\times 10^{-12}$	$333333333333\times 10^{-12}$
1	0.10	$220833333333\times 10^{-12}$	$212282986111\times 10^{-12}$	$212176886571\times 10^{-12}$

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2	0.20	174270833333$\times 10^{-12}$	162765457718$\times 10^{-12}$	162626480390$\times 10^{-12}$
3	0.30	176419270833$\times 10^{-12}$	164516540751$\times 10^{-12}$	164376720049$\times 10^{-12}$
4	0.40	216512044271$\times 10^{-12}$	205240505195$\times 10^{-12}$	205111761079$\times 10^{-12}$
5	0.50	287820027669$\times 10^{-12}$	277476660704$\times 10^{-12}$	277361666208$\times 10^{-12}$
6	0.60	386137517293$\times 10^{-12}$	376698077980$\times 10^{-12}$	376595689456$\times 10^{-12}$
7	0.70	508835948308$\times 10^{-12}$	500157948357$\times 10^{-12}$	500065794474$\times 10^{-12}$
8	0.80	654272467693$\times 10^{-12}$	646189588456$\times 10^{-12}$	646105212963$\times 10^{-12}$
9	0.90	821420292308$\times 10^{-12}$	813781703412$\times 10^{-12}$	813702998846$\times 10^{-12}$
10	1.00	1009637682692$\times 10^{-12}$	1002320668998$\times 10^{-12}$	1002245982333$\times 10^{-12}$

Table 2: The proposed techniques Error estimation are compared to exact answer

n	**t_n**	**Modified Euler' Technique**	**Runge – Kutta Technique of order 4th**	**Exact Solution**
0	0.00	0.000000000000	0.000000000000	333333333333$\times 10^{-12}$
1	0.10	8656446762E−12	10609954 E−12	212176886571$\times 10^{-12}$
2	0.20	1164435294E−12	13897732 E−12	162626480390$\times 10^{-12}$
3	0.30	1204255078E−12	13982070 E−12	164376720049$\times 10^{-12}$
4	0.40	1140028319E−12	12874411E−12	205111761079$\times 10^{-12}$
5	0.50	1045836146E−12	1149944 E−12	277361666208$\times 10^{-12}$
6	0.60	954182783 E−12	10238852 E−12	376595689456$\times 10^{-12}$
7	0.70	8770153834E−12	92153883 E−12	500065794474$\times 10^{-12}$
8	0.80	8167254730E−12	84375494 E−12	646105212963$\times 10^{-12}$
9	0.90	771729346 E−12	78704566 E−12	813702998846$\times 10^{-12}$
10	1.00	7391700359E−12	74686665 E−12	1002245982333$\times 10^{-12}$

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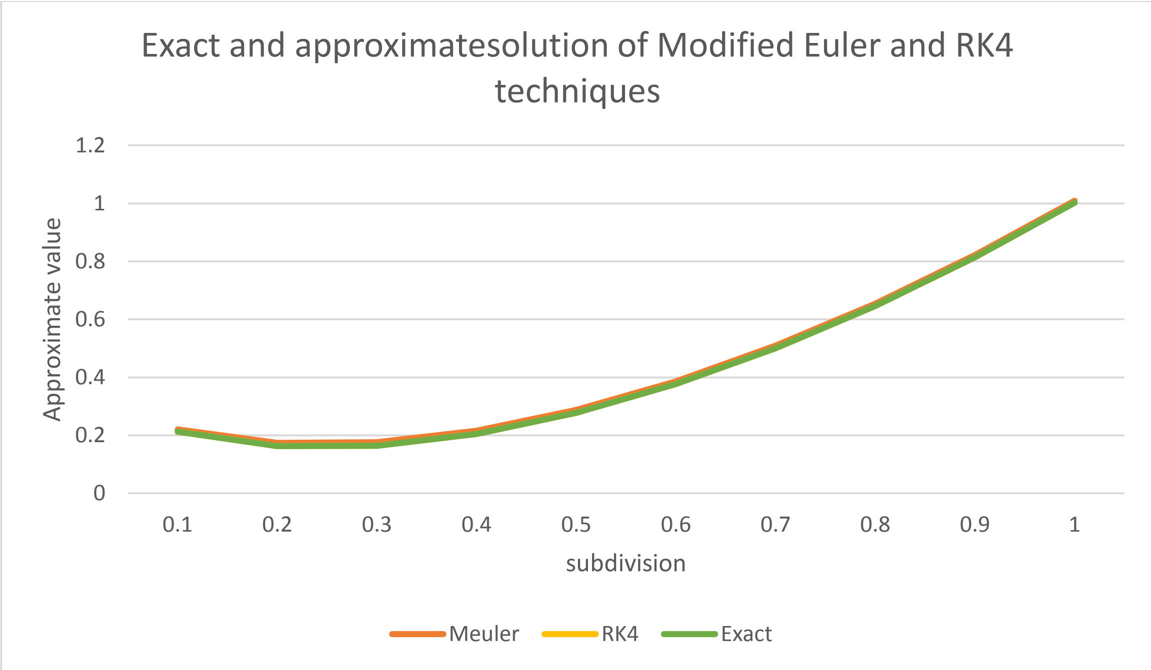

Figure1: The above figure represents Approximate solution for proposed techniques with exact solution.

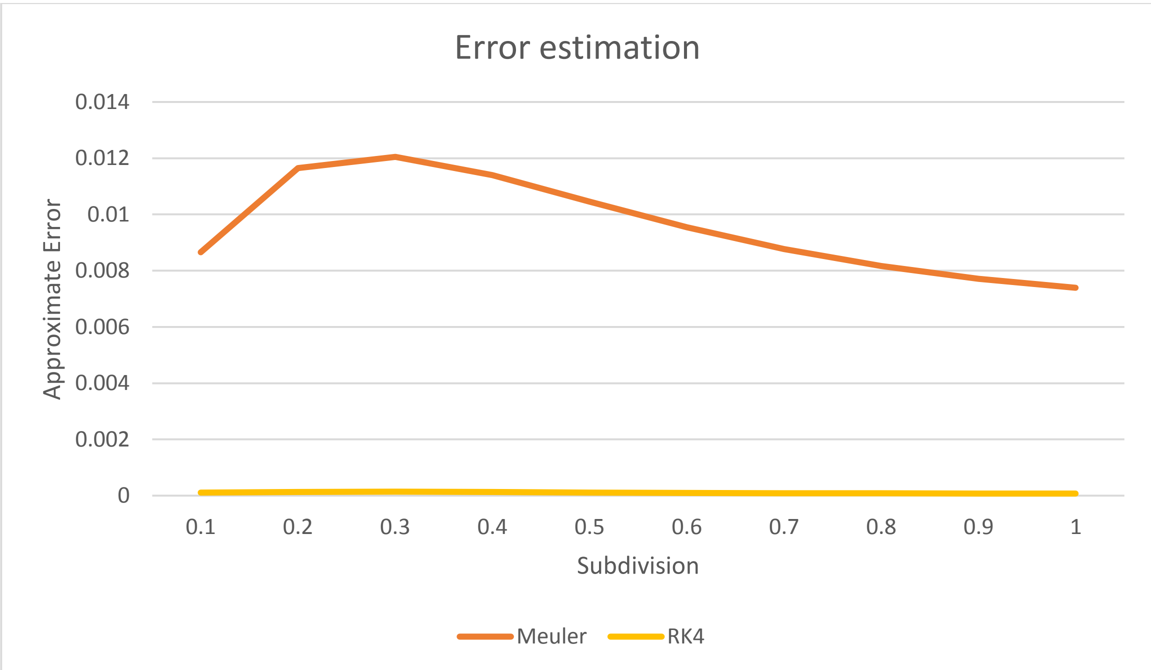

Figure 2: Error graph for proposed two techniques.

Example (2)

Let's assume differential equation $\frac{dy}{dt} = \frac{2}{t^2+1} - \frac{2ty}{t^2+1}$ in interval $0 \leq t \leq 1$ with y(0) = 1.

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Exact solution of above differential equation is y(t) = $\frac{2t+1}{t^2+1}$

Now, using proposed techniques and h = 0.1, obtained approximate solution of above differential equation, then estimate error estimation of proposed techniques & lastly draw graphs of solutions of above differential equation as shown below.

Table 3: Approximate solution of proposed Techniques.

n	**t_n**	**Modified Euler's Technique**	**Runge - Kutta Technique of order 4^{th}**	**Exact solution**
0	0.00	1000000000×10^{-9}	1000000000×10^{-9}	1000000000×10^{-9}
1	0.10	1187128712×10^{-9}	1188118764×10^{-9}	1188118811×10^{-9}
2	0.20	1344353306×10^{-9}	1346153608×10^{-9}	1346153846×10^{-9}
3	0.30	1465526995×10^{-9}	1467889340×10^{-9}	1467889908×10^{-9}
4	0.40	1549060650×10^{-9}	1551723173×10^{-9}	1551724137×10^{-9}
5	0.50	1597265955×10^{-9}	1599998664×10^{-9}	1600000000×10^{-9}
6	0.60	1615015698×10^{-9}	1617645444×10^{-9}	1617647058×10^{-9}
7	0.70	1608321183×10^{-9}	1610736481×10^{-9}	1610738255×10^{-9}
8	0.80	1583221347×10^{-9}	1585364030×10^{-9}	1585365853×10^{-9}
9	0.90	1545108042×10^{-9}	1546959536×10^{-9}	1546961325×10^{-9}
10	1.00	1498430678×10^{-9}	1499998300×10^{-9}	1500000000×10^{-9}

Table 4: The proposed two approaches Error estimation are compared to exact answer.

n	**t_n**	**Modified Euler's Technique**	**Runge – Kutta Technique of order 4^{th}**	**Exact Solution**
0	0.00	0.000000000	0.000000000	1000000000×10^{-9}
1	0.10	990099 E−09	47 E−09	1188118811×10^{-9}
2	0.20	1800539 E−09	237 E−09	1346153846×10^{-9}
3	0.30	2362908 E−09	567 E−09	1467889908×10^{-9}
4	0.40	2663487 E−09	964 E−09	1551724137×10^{-9}
5	0.50	2734044 E−09	1335 E−09	1600000000×10^{-9}
6	0.60	2631359 E−09	1613 E−09	1617647058×10^{-9}
7	0.70	2417071 E−09	1773 E−09	1610738255×10^{-9}
8	0.80	2144506 E−09	1823 E−09	1585365853×10^{-9}
9	0.90	1853283 E−09	1789 E−09	1546961325×10^{-9}
10	1.00	1569321 E−09	1699 E−09	1500000000×10^{-9}

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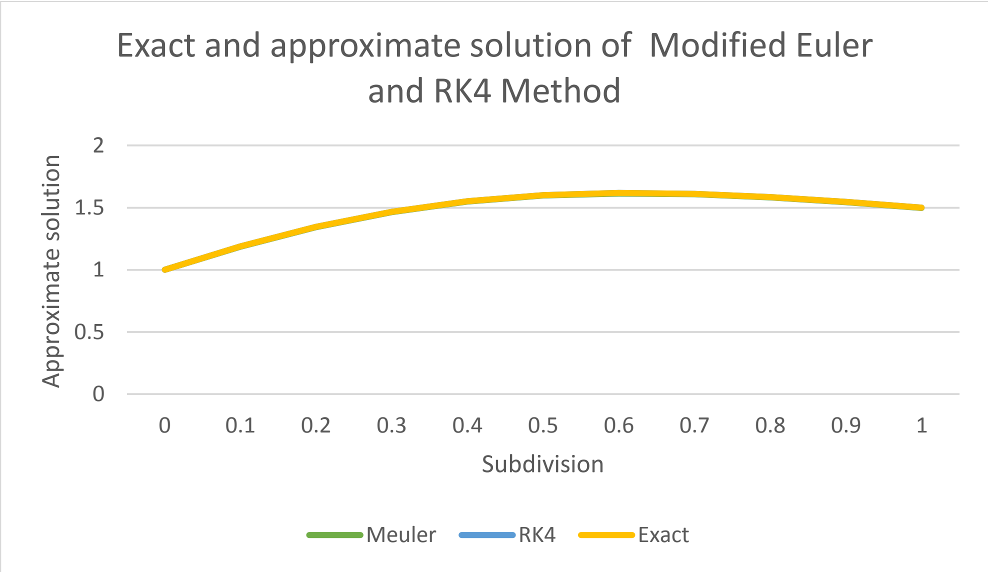

Figure 3: The above figure represents Approximate solution for proposed techniques with actual solution.

Figure 4: Error graph for proposed techniques.

2. RESULT DISCUSSION

Tables 1 and 3 show roughly acquired answers from two examples, while Tables 2 & 4 show two approaches' error analyses when compared to precise solution. Figures 1 and 3 show the two cases' approximations of the results. Figures 2 and 4 alongside each other to show graphically depict the error estimation of the suggested method for two examples. According to results table, Runge-Kutta fourth order technique converges more frequently to exact

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solution than Modified Euler techniques. The amount of error is largest for the modified Euler technique and minimal for Runge-Kutta fourth technique of fourth order, according to error graphs. In summary, we obtained that Runge-Kutta fourth order technique is most successful technique for solving IVPs of ordinary differential equations since it converges precisely & quickly in comparison to the Modified Euler methods.

3. CONCLUSION

This study employs Runge-Kutta, Modified Euler techniques to solve initial value issue of differential equation, & assesses degree of correctness of each of these suggested approaches. The error table and error graphs provide a clear picture of which strategy is more effective. Even if the numerical answers of all techniques are in good agreement with actual answer, Runge-Kutta 4^{th} order methodology was shown to more reliable & to converge to exact solution more quickly than Modified Euler methods. The Runge-Kutta technique is frequently used because it is believed to be more precise, stable, consistent, & convergent than any other techniques that have been proposed.

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