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The Connected Geodetic Global Domination Number of a Graph

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Abstract: A set S of vertices in a connected graph {G=(V,E)} is called a geodetic set if every vertex not in S lies on a shortest path between two vertices from S. A set D of vertices in G is called a dominating set of G if every vertex not in D has at least one neighbour in D. A geodetic dominating set S is both a geodetic and a dominating set. A set S is called a geodetic global dominating set of G if S is both geodetic and global dominating set of G. The geodetic global domination number (geodetic domination number) is the minimum cardinality of a geodetic global dominating set (geodetic dominating set) in G. In this paper we introduced and investigate the connected geodetic global domination number of certain graphs and some of the general properties are studied.

Keywords: Geodetic Set, Dominating Set, Geodetic Global Domination, Connected Geodetic Global Domination.

1. INTRODUCTION

By a graph $G = (V,E)$ we mean a finite, connected, undirected graph with neither loopN nor multiple edges. The *order* $|V|$ and *size* $|E|$ of G and denoted by p and q
respectively. For graph theoretic terminology we refer to west [9]. The *open neighbourhood* of any vertex v in G is $N(v) = \{x : xv \in E(G)\}$ and *closed neighbourhood* of a vertex v in G is $N[v] = N(v)\cup\{v\}$. The *degree* of a vertex in the graph G is denoted by *deg(v)* and the maximum degree (minimum degree) in the graph G is denoted by $4(G)(\delta(G))$. For a set $S \subseteq V(G)$ the open (closed) neighbourhood $N(S)(N[S])$ in G is defined as $N(S) = \bigcup_{V\in s} N(v)(N[S] = \bigcup_{v\in S} N[v])$.

If G is a connected graph the *distance* $d(x,y)$ is the length of a shortest x-y path in G. The *diameter* is defined by $diam(G) = max_{x,y\in V(G)} d(x,y)$. Two vertices u and v are said to be *antipodal vertices* if $d(u,v) = diam(G)$. If $e = \{u,v\}$ is an edge of a graph G with $deg(u) = 1$ and $deg(v) > 1$, then we call e a pendant edge, u a pendant vertex and v a support vertex. A vertex v of G is said to be an *extreme vertex* of the subgraph induced by its neighbourhood is

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complete. The set of all extreme vertices is denoted by *Ext*(G). A vertex v is said to be *full vertex* if v is adjacent to all other vertices in G, that is, $deg(v) = p-1$. The set of all full vertices is denoted by *Fx*(G). A cut vertex of G is a vertex whose removal results a disconnected graph. The set of all cut vertices of G is denoted by *Cut*(G). The *girth* of a graph G is the length of a shortest cycle contained in G and is denoted by $c(G)$. An acyclic connected graph is called a *tree*. A set $S \subseteq V$ is said to be *clique* in G if $\langle S \rangle$ is complete. An x-y path of length $d(x,y)$ is called *geodesic*. A vertex v is said to lie on an geodesic P if v is an internal vertex of P. The closed interval consists of x,y and all vertices lying on some x-y geodesic of G and for a non empty set $S \subseteq V(G)$, $I[S] = \bigcup_{x,y \in S} I[x, y]$.

A set $S \subseteq V(G)$ in a connected graph is a *geodetic set* of G if $I[S] = V(G)$. The *geodetic number* of G, denoted by $g(G)$, is the minimum cardinality of a geodetic set of G. The geodetic number of a disconnected graph is the sum of the geodetic number of its components. A geodetic set of cardinality $g(G)$ is called $g(G)$-set. Various concepts inspired by geodetic sets are introduced in [1, 2].

A set $S \subseteq V(G)$ in a graph G is a *dominating set* of G if for every vertex v in $V - S$, there exists a vertex $u \in S$ such that v is adjacent to u. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G [5].

The complement $\overline{G} = (\overline{V}, \overline{E})$ of $G = (V,E)$ is the graph with the vertex set $\overline{V} = \{\overline{v} : v \in V\}$ and $E = \{(\overline{u}, \overline{v}) : \overline{u}, \overline{v} \in \overline{V}, u \notin v$ and $(u,v) \notin E\}$. The domination number of $\overline{G}$ is denoted by $\gamma(G)$.

A set $S \subseteq V(G)$ is called a *global dominating set* of $G = (V,E)$ if it is a dominating set of both G and $\overline{G}$ [8]. The *global domination number* of G, denoted by $\overline{\gamma(G)}$, is the minimum cardinality of a global dominating set of G.

In [7], authors studied geodetic global domination in graphs which is defined as follows: Let $G = (V,E)$ be a connected graph. A subset $S \subseteq V(G)$ is called *geodetic global dominating set* of G if S is both geodetic and global dominating set of G. The *geodetic global domination number* denoted by $\overline{\gamma_g}(G)$ is the minimum cardinality of a geodetic global dominating set of G and the geodetic global dominating set with cardinality $\overline{\gamma_g}(G)$ is called the $\overline{\gamma_g}$-set of G or $\overline{\gamma_g}(G)$-set.

In this paper we introduce a new parameter known as connected geodetic global domination number of a graph.

2. SOME USEFUL KNOWN RESULTS

Theorem 2.1. *[7] Let G be a connected graph. Then Ext(G) ⊆ S for all geodetic global dominating sets S in G.*

Theorem 2.2. *[7] Let G be a connected graph. Then Fx(G) ⊆ S for all geodetic global dominating sets S in G.*

Theorem 2.3. *[7] For any connected graph G with cut vertex v, every geodetic global dominating set contains at least one vertex from each component of G − {v}.*

Theorem 2.4. *Let G be a connected graph and let S ⊆ V(G) be a clique in G. Then S is a geodetic global dominating set of G if and only if S = V(G).*

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Theorem 2.5. *[7] Let G be a connected graph of order p. Then,* $\overline{\gamma_g}(G)$ *= 2 if and only if G = K_2 or there exists a geodetic set S = {u,v} such that d(u,v) = 3.*

Theorem 2.6. *[7] Let G be a connected graph of order p. Then,* $\overline{\gamma_g}(G)$ *= p if and only if G contains only the extreme and full vertices.*

Connected Geodetic Global Domination Number Of A Graph

Definition 3.1. Let G be a connected graph with at least two vertices. A connected geodetic global dominating set S of G is a geodetic global dominating set such that the subgraph $\langle S\rangle$ induced by S is connected. The minimum cardinality of a connected geodetic global dominating set of G is the connected geodetic global domination number of G and is denoted by $\overline{\gamma_{gc}}(G)$. A connected geodetic global dominating set of cardinality $\overline{\gamma_{gc}}(G)$ is called a $\overline{\gamma_{gc}}$-set of G or $\overline{\gamma_{gc}}(G)$- set.

Example 3.2. For the graph G in Figure 3.1, it is easy to verify that $S = \{v_1,v_3,v_4\}$ is a minimum geodetic global dominating set of G and so $\overline{\gamma_g}(G) = 3$. Since $\langle S\rangle$ is not connected, that S is not a connected geodetic global dominating set of G.

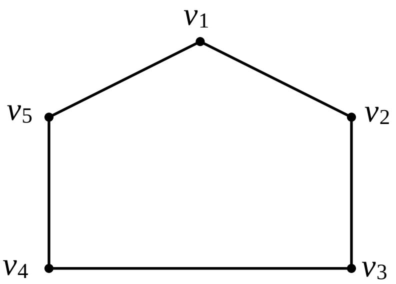

Figure 3.1

Now, consider $S_1 = \{v_1,v_2,v_3,v_4\}$. It is clear that S_1 is a minimum connected geodetic global dominating set in G and so $\overline{\gamma_{gc}}(G) = 4$. Thus, the geodetic global domination number and connected geodetic global domination number are different.

Remark 3.3. Minimum connected geodetic global dominating set of a connected graph need not be unique. From the graph G in Figure 3.1, $S_1 = \{v_1,v_2,v_3,v_4\}$, $S_2 = \{v_2,v_3,v_4,v_5\}$, $S_3 = \{v_1,v_3,v_4,v_5\}$, $S_4 = \{v_1,v_2,v_4,v_5\}$ and $S_5 = \{v_1,v_2,v_3,v_5\}$ are the minimum connected geodetic global dominating sets in G. Thus there can be more then one $\overline{\gamma_{gc}}(G)$-set for a connected graph.

Remark 3.4. Every connected geodetic global dominating set is a geodetic global dominating set, but the converse need not be true. For the graph G given in Figure 3.1, the set $S = \{v_1,v_3,v_4\}$ is a geodetic global dominating set but not a connected geodetic global dominating set of G.

Theorem 3.5. *Every connected geodetic global dominating set of a connected graph G contains all its extreme vertices. In fact, if the set S of all extreme vertices of G is a connected geodetic global dominating set, then S is the unique minimum connected geodetic global dominating set of G .*

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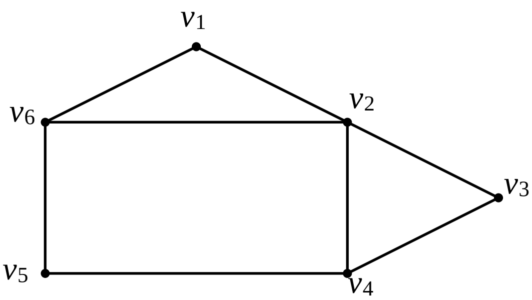
Proof. Since every connected geodetic global dominating set of G also a geodetic global dominating set of G, the result follows from Theorem 2.1. □

Corollary 3.6. *For any complete graph $K_p(p \geq 2)$, $\overline{\gamma}_{gc}(G) = \overline{\gamma}_g(G) = p$.*

The following theorem follows directly from the definition.

Theorem 3.7. *For any connected graph G of order $p \geq 2$, $2 \leq \overline{\gamma}_g(G) \leq \overline{\gamma}_{gc}(G) \leq p$.*

Remark 3.8. The bound for $\overline{\gamma}_{gc}(G)$ in Theorem 3.7 are sharp. Consider $K_p(p \geq 2)$. By Corollary 3.6, $\overline{\gamma}_g(K_p) = \overline{\gamma}_{gc}(K_p) = p$. Also all the inequalities in Theorem 3.7 are strict. For the graph G given in Figure 3.2, $S = \{v_1,v_3,v_5\}$ is a $\overline{\gamma}_g$-set of G and $S_1 = \{v_1,v_2,v_3,v_4,v_5\}$ is a $\overline{\gamma}_{gc}$-set of G. Also, $|V(G)| = 6$. Therefore, $p = 6$, $\overline{\gamma}_g(G) = 3$ and $\overline{\gamma}_{gc}(G) = 5$. Thus, $2 < \overline{\gamma}_g(G) < \overline{\gamma}_{gc}(G) < p$.



G

Figure 3.2

Corollary 3.9. *Let G be any connected graph. If $\overline{\gamma}_{gc}(G) = 2$, then $\overline{\gamma}_g(G) = 2$.*

Corollary 3.10. *Let G be any connected graph. If $\overline{\gamma}_g(G) = p$, then $\overline{\gamma}_{gc}(G) = p$.*

Corollary 3.11. *Let G be any connected graph with $|Ext(G)| = n$. Then, $\overline{\gamma}_{gc}(G) \geq max\{2,n\}$.*

Proof. This follows from Theorem 3.5 and 3.7. □

Theorem 3.12. *Let G be a connected graph with cut vertices and let S be a connected geodetic global dominating set of G. If v is a cut vertex of G, then every component of $G - v$ contains at least one element of S.*

Proof. Let v be a cut vertex of G and S be a connected geodetic global dominating set of G. Since S is also a geodetic global dominating set of G, by Theorem 2.3, every component of $G - v$ contains at least one element of S □

Corollary 3.13. *Let G be a connected graph with cut vertices and let S be a connected geodetic global dominating set of G. Then, every branch of G contains an element of S*

Proof. Let S be a connected geodetic global dominating set of G and v be a cut vertex of G. Since every branch of G at v is a component of $G-v$ with the vertex v together all the edges joining at v to $V(H)$. Therefore, by Theorem 3.12, we conclude every branch of G contains an element of S. □

Theorem 3.14. *Each full vertex and cut vertex of a connected graph G belongs to every connected geodetic global dominating set of G.*

Proof. Since every connected geodetic global dominating set is a geodetic global dominating set by Theorem 2.2, each full vertex belongs to every connected geodetic global dominating set of G. Now, let S be a connected geodetic global dominating set of G and let v be a cut vertex of G. Then, take $G_1,G_2,...,G_n(n \geq 2)$ be the components of $G - v$. By Theorem 3.12, S

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contains atleast one vertex from each $G_1, G_2, \ldots, G_n$. Since every vertices in G is connected, it follows that $v \in S$. □

Corollary 3.15. *For any non-complete connected graph G with $|Ext(G)| = k$ and $|Fx(G)| = l$, $\overline{\gamma_{gc}}(G) \geq max\{2, k + l\}$.*

Proof. This follows from Theorem 3.5 and Theorem 3.14. □

Corollary 3.16. *For the star graph $K_{1,p-1}(p \geq 2)$, $\overline{\gamma_{gc}}(K_{1,p-1}) = p$.*

Proof. This follows from Theorem 3.5 and Theorem 3.14. □

Theorem 3.17. *Let G be a connected graph of order $p \geq 2$. Then, $\overline{\gamma_{gc}}(G) = p$ if and only if every vertex of G is either a full vertex or a cut vertex or an extreme vertex.*

Proof. Assume $\overline{\gamma_{gc}}(G) = p$. To prove every vertex of G is either a full vertex or a cut vertex or an extreme vertex. Suppose G contains a vertex v which is not a full or cut or extreme vertex. Since v is not an extreme vertex, that $G \neq K_p$ and so $V(G) - \{v\}$ is a geodetic set of G. Also, v is not a full vertex implies *that* $G \neq K_p$.. Since G is connected, $V(G) - \{v\}$ is a global dominating set of G. Moreover, v is not a cut vertex of G, $\langle V(G) - \{v\} \rangle$ is connected. Therefore, $V(G) - \{v\}$ is a connected geodetic global dominating set of G and so $\overline{\gamma_{gc}}(G) \leq |V(G) - \{v\}| = p - 1$, which is a contradiction. Conversely, assume every vertex of G is either a full vertex or a cut vertex or an extreme vertex. If $G = K_p$, then by Corollary 3.6, $\overline{\gamma_{gc}}(G) = p$. If G 6= K_p, then the result follows from Theorem 3.7 and Corollary 3.15. □

Theorem 3.18. *For any non-trivial tree T of order p, $\overline{\gamma_{gc}}(T) = p$.*

Proof. Let T be a non-trivial tree of order p. Then, it is clear that T contains only the extreme and cut vertices. Hence, by Theorem 3.17, $\overline{\gamma_{gc}}(T) = p$. □

Corollary 3.19. *For any path graph $P_p(p \geq 2)$, $\overline{\gamma_{gc}}(P_p) = p$.*

Theorem 3.20. *If G is a connected graph and $S \subseteq V(G)$ is a clique in G, then S is a connected geodetic global dominating set of G if and only if $S = V(G)$.*

Proof. Let G be a connected graph and $S \subseteq V(G)$ be a clique in G. Since every connected geodetic global dominating set in G is a geodetic global dominating set of G, the result follows Theorem 2.4. □

Theorem 3.21. *Let G be a connected graph of order p. Then, $\overline{\gamma_{gc}}(G) \geq 3$ if and only if $p \geq 3$.*

Proof. Let G be a connected graph of order p. First assume $p \geq 3$. We show that $\overline{\gamma_{gc}}(G) \geq 3$. Suppose on the contrary $\overline{\gamma_{gc}}(G) < 3$. By Theorem 3.7, the only possible is $\overline{\gamma_{gc}}(G) = 2$. Let $S = \{x, y\}$ be a $\overline{\gamma_{gc}}$-set of G. Then, by definition $\langle S \rangle$ is connected, and so xy is an edge in G. Also, there is no other vertex lies on the geodesic joining pair of vertices of S. Therefore, S is a clique in G and so by Theorem 3.20, $|V(G)| = |S| = 2$. That is, the order of G is $p = 2$, which is a contradiction. Hence, $\overline{\gamma_{gc}}(G) \geq 3$. The converse part is obvious. □

Corollary 3.22. *Let G be a connected graph of order $p \geq 2$. Then, $\overline{\gamma_{gc}}(G) = 2$ if and only if $G = K_2$.*

Theorem 3.23. *Let G be a connected graph of order $p \geq 3$. Then, $\overline{\gamma_{gc}}(G) = 3$ if and*

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only if $G = K_3$ *or* $G \approx \overline{K_2} + H$, *where H is any graph of order* $p - 2$ *and* $\delta_H(v) = 0$ *for some* $v \in V(H)$.

Proof. Let G be a connected graph of order $p \geq 3$. If $G = K_3$, then by Corollary 3.6, $\overline{\gamma_{gc}}(G) = 3$. Suppose $G \approx \overline{K_2} + H$ where H is a graph of order $p-2$ and $\delta(v) = 0$ for some $v \in V(H)$. Let $V(K_2) = \{x,y\}$. Then every vertex in G lies on $x - y$ geodesic and so $\{x,y\}$ is a geodetic set in G. But $\{x,y\}$ is not a global dominating set in G. Since $\delta(v) = 0$ in H, v is adjacent to every vertices in $\overline{H}$ and so $S = \{v,x,y\}$ is a dominating set in G. Also, S dominates G. It follows that S is a global dominating set as well as geodesic set in G. Since $\langle S \rangle$ is connected, S is a connected geodetic global dominating set in G and so $\overline{\gamma_{gc}}(G) \leq |S| = 3$. Since $p \geq 3$ by Theorem 3.21, we conclude that $\overline{\gamma_{gc}}(G) = 3$.

Conversely, assume that $\overline{\gamma_{gc}}(G) = 3$. Let $S = \{x,y,z\}$ be a $\overline{\gamma_{gc}}(G)$ set . Since, that $\langle S \rangle$ is connected, there exist vertices in S, which is adjacent to every vertices in S. Without loss of generality, we assume that x is adjacent to both y and z. If y and z are adjacent in G, then $G = K_3$ and we are done. Now, y and z are non-adjacent in $\bar{G}$. Then, each vertex not in S must lie on a geodesic joining y and z. It follows that $d(y,z) = 2$ in G. and so $G \approx \overline{K_2} + H$ where $V(K_2) = \{y,z\}$ and H is a graph of order $p-2$. Now, we show that $\delta_H(v) = 0$ for some $v \in V(H)$. Suppose $\delta_H(v) \geq 1$ for every $v \in V(H)$. Then v is adjacent to at least one vertex in H and so S is not a global dominating set of G. This shows that S is not a connected geodetic global dominating set of G, which is a contradiction. Hence the result is over. □

Theorem 3.24. *Let G be a connected graph of order* $p \geq 3$. *If* $\overline{\gamma_g}(G) = 2$, *then* $\overline{\gamma_{gc}}(G) = 4$.

Proof. Let G be a connected graph of order $p \geq 3$ and let $\overline{\gamma_g}(G) = 2$. Then by Theorem 2.5, $G = K_2$ or G has a geodetic set $S = \{x,y\}$ with $d(x,y) = 3$. Since $p \geq 3$, we have G 6= K_2. By hypothesis, S is a geodetic global dominating set of G. But $\langle S \rangle$ is not connected. Let P be a $x - y$ geodesic path in G. Since $d(x,y) = 3$ *in* G, $S' = V(P)$ is a connected geodetic global dominating set of G and so $\overline{\gamma_{gc}}(G) \leq |S'| = diam(G) + 1$. Thus, $\overline{\gamma_{gc}}(G) \leq 4$. Since $p \geq 2$, by Theorem 3.21, $\overline{\gamma_{gc}}(G) \geq 3$. Suppose $\overline{\gamma_{gc}}(G) = 3$. Then by Theorem 3.23, $G = K_3$ or $G \approx \overline{K_2} + H$ where H is any graph of order $p - 2$ and $\delta_H(v) = 0$ for some $v \in V(H)$. If $G = K_3$,

then by Corollary 3.6, $\overline{\gamma_g}(G) = 3$, a contradiction. If $G \approx \overline{K_2} + H$, then $diam(G) = 2$. Therefore, by Theorem 2.5, $\overline{\gamma_g}(G) \geq 3$, again a contradiction. Hence, $\overline{\gamma_{gc}}(G) = 4$.□

Remark 3.25. *The converse of the Theorem 3.24, need not be true. Consider the graph G in Figure 3.3 It is clear that G is a connected graph with* $\overline{\gamma_{gc}}(G) = 4$. *But* $\overline{\gamma_g}(G) \neq 2$.

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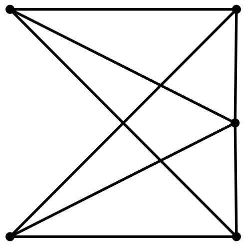

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Figure 3.3

Theorem 3.26. *For the complete bipartite graph* $K_{p,q}(p,q \geq 2)$, $\overline{\gamma_{gc}}(G) = min\{p,q\} + 1$ if $2 \leq p,q \leq 3$ and $\overline{\gamma_{gc}}(G) = 4$ if $p,q \geq 4$.

Proof. Let $G = K_{p,q}$. Let the bipartitions of G be $X = \{x_1,x_2,...,x_p\}$ and $Y = \{y_1,y_2,...,y_q\}$. Consider $2 \leq p,q \leq 3$. First assume $p < q$. Then $X = \{x_1,x_2\}, Y = \{y_1,y_2,y_3\}$. By Theorem 3.21, $\overline{\gamma_{gc}}(G) \geq 3$. On the other hand $S = X \cup \{y_i\}$, where $1 \leq i \leq 3$ be a connected geodetic global dominating set of G and so $\overline{\gamma_{gc}}(G) \leq |S| = 3$. Thus, $\overline{\gamma_{gc}}(G) = 3$. Similarly, we can prove for $p = q$. Therefore $\overline{\gamma_{gc}}(G) = min\{p,q\} + 1$.

Now, consider $p,q \geq 4$. Let $S = \{x_i,x_{i+1},y_j,y_{j+1}\}$ where $1 \leq i \leq p, 1 \leq j \leq q$. Since, each $x_i - x_{i+1}$ geodesic path contains all vertices of Y as an internal vertices and each $y_j - y_{j+1}$ geodesic path contains all vertices of X as an internal vertices, that S is a geodetic set of G. Since, $\{x_i,y_i\}$ dominates G as well as $\bar{G}$, that S is a global dominating set of G. Moreover, since $< S >$ is connected, S is a connected geodetic global dominating set of G and so $\overline{\gamma_{gc}}(G) \leq |S| = 4$. Since, $|X| \geq 4$ and $|Y| \geq 4$ by Theorem 3.21 and Theorem 3.23, we have $\overline{\gamma_{gc}}(G) \geq 4$. Hence $\overline{\gamma_{gc}}(G) = 4$. □

Theorem 3.27. *For the cycle graph* $C_p(p \geq 4)$, $\overline{\gamma_{gc}}(G) = p - 1$ if p=4 or 5 and $\overline{\gamma_{gc}}(G) = p - 2$ if $p \geq 6$..

Proof. Let $G = C_p$ and $V(C_p) = \{v_1,v_2,...,v_p\}$, where $p \geq 4$.If $p = 4$, then $G = C_4$ and so by Theorem 3.23, $\overline{\gamma_{gc}}(G) = 3 = p-1$. If $p = 5$, then $G = C_5$ by Theorem 3.21 and Theorem 3.23, $\overline{\gamma_{gc}}(G) \geq 4$. Suppose $\overline{\gamma_{gc}}(G) = 5$, then by Theorem 3.17 every vertex of G is either a full vertex or extreme vertex or a cut vertex. Since, G is not a path graph, G contains at least one vertex of degree 3, which is a contradiction to every vertex of G is of degree 2. Thus, $\overline{\gamma_{gc}}(G) = 4 = p - 1$.

Now, let $p \geq 6$. We consider three cases:

Case (i) $p \equiv 0 (\text{mod } 3)$. Let $S = \{v_1,v_2,v_5,...,v_{p-4},v_{p-1}\}$. It is clear that S is a geodetic global dominating set of G. But $\langle S \rangle$ is not connected. Take $S_1 = \{v_1,v_2,v_3,v_4,v_5,...,v_{p-4}\} \cup \{v_{p-1},v_p\}$. Clearly S_1 is a minimum connected geodetic global dominating set of G and so $\overline{\gamma_{gc}}(G) = |S_1| = p - 2$.

Case (ii) $p \equiv 1 (\text{mod } 3)$. let $S' = \{v_1,v_2,v_5,...,v_{p-5},v_{p-2}\}$. It is clear that S' is a geodetic global dominating set of G. But $\langle S' \rangle$ is not connected. Take

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$S_2 = \{v_1,v_2,v_3,v_4,v_5,...,v_{p-5}\} \cup \{v_{p-2},v_{p-1},v_p\}$. Clearly S_2 is a minimum connected geodetic global dominating set of G and so $\overline{\gamma_{gc}}(G) = |S_2| = p - 2$.

Case (iii) $p \equiv 2 (\text{mod } 3)$. Let $S'' = \{v_1,v_2,v_5,...,v_{p-3},v_p\}$. One can easily verified that S'' is a geodetic global dominating set of G. But $\langle S'' \rangle$ is not connected.

Take $S_3 = \{v_1,v_2,v_3,v_4,v_5,...,v_{p-3}\} \cup \{v_p\}$. Clearly, S_3 is a minimum connected geodetic global dominating set of G and so $\overline{\gamma_{gc}}(G) = |S_3| = p - 2$. Hence, all the cases $\overline{\gamma_{gc}}(G) = p - 2$. □

Theorem 3.28. *Let G be a connected graph of order p. If* $Fx(G) \neq \varphi$. *Then* $\overline{\gamma_{gc}}(G) = \overline{\gamma_g}(G)$.

Proof. Let G be a connected graph of order p with $Fx(G) \neq \varphi$. Let S be a $\overline{\gamma_{gc}}(G)$-set. Then by Theorem 2.2, $Fx(G) \subseteq S$. Also, for every $v \in Fx(G)$, v is adjacent for every vertices in S. it follows that $< S >$ is connected. Therefore, S is a connected geodetic global dominating set of G and so $\overline{\gamma_{gc}}(G) = \overline{\gamma_g}(G). \leq |S| = \overline{\gamma_g}(G)$. By Theorem 3.7, we conclude, $\overline{\gamma_{gc}}(G) = \overline{\gamma_g}(G)$.

□

Remark 3.29. *The converse of Theorem 3.28 need not be true. For the graph G given in Figure 3.4, it is clear that* $S = \{v_1,v_2,v_3,v_4,v_7\}$ *is the unique* $\overline{\gamma_g}(G)$*-set as well as* $\overline{\gamma_{gc}}(G)$*-set and so* $\overline{\gamma_{gc}}(G) = \overline{\gamma_g}(G) = 5$. *But,* $Fx(G) = \varphi$.

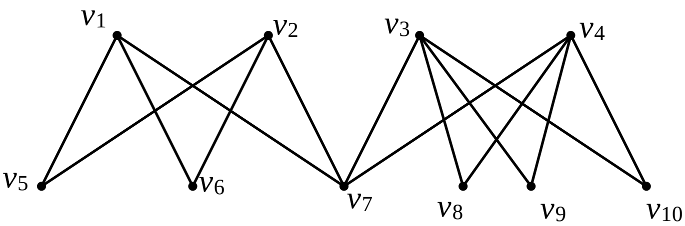

Figure 3.4

Theorem 3.30. *For every pair of integers a, p with* $3 \leq a \leq p$, *there exists a connected graph G of order p with* $\overline{\gamma_{gc}}(G) = a$.

Proof. We consider two cases:

Case (i) $3 \leq a = p$. If $G = K_p$, then by Corollary 3.6, $\overline{\gamma_{gc}}(G) = a$ or if G is a tree of order p, then by Theorem 3.18, $\overline{\gamma_{gc}}(G) = a$.

Case (ii) $3 \leq a < p$. let $P_{a-2} : x_1,x_2,...,x_{a-2}$ be a path of order $a - 2$. Consider the graph G obtained from P_{a-2} and $K_{2,p-a+1}$ with partite sets $\{x,x_1\}$ and $\{y_1,y_2,...,y_{p-a+1}\}$ as shown in Figure 3.5. Clearly, G is a connected graph of order p.

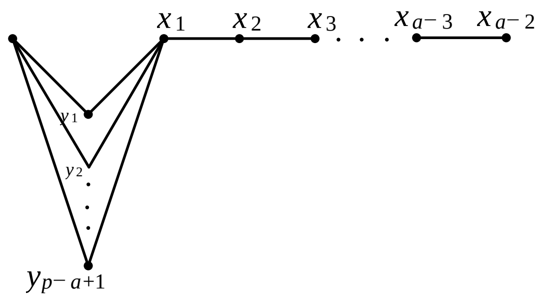

Figure 3.5

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Also, $Ext(G) = \{x_{a-2}\}$ and $Cut(G) = \{x_1,x_2,...,x_{a-3}\}$. Let $S = Ext(G) \cup Cut(G)$. By Theorem 3.5 and Theorem 3.14, every connected geodetic global dominating set contains S. It is easily verified that S is not a connected geodetic global dominating set of G and so $\overline{\gamma_{gc}}(G) > |S| = a - 2$. Also, for any $v \notin S$, $S \cup \{v\}$ is not a connected geodetic global dominating set of G. It is clear that, $S \cup \{x,y_i\}$ where, $1 \le i \le p - a + 1$ is a connected geodetic global dominating set of G and so $\overline{\gamma_{gc}}(G) = a$. □

Theorem 3.31. *For three integers r,t and $a \ge d + 1$ with $1 \le r \le d \le 2r$, there exists a connected graph G such that $rad(G) = r$, $diam(G) = d$ and $\overline{\gamma_{gc}}(G) = a$.*

Proof. If $r = 1$ then $d = 1$ or 2. If $d = 1$, then pick $G = K_a$ and so by Corollary 3.6, $\overline{\gamma_{gc}}(G) = a$. If $d = 2$ then we take $G = K_{1,a-1}$ and so by Corollary 3.15, $\overline{\gamma_{gc}}(G) = a$. Now assume $r \ge 2$. We prove by considering two cases:

Case (i) $r = d$. First assume $a = d+1$. Consider G as a path graph of order of a. Then by Corollary 3.19, $\overline{\gamma_{gc}}(G) = a$. Also, it is clear that $diam(G) = rad(G) = a - 1 = d$. Now, assume $a \ge d + 2$. Let $C_{2r} : y_1,y_2,...,y_{2r},y_1$ be the even cycle of order $2r$ and let K_{a-2r+2} be a complete graph with $V(K_{a-2r+2}) = \{x_1,x_2,...,x_{a-2r+2}\}$. Let G be the graph obtained from K_{a-2r+2} and C_{2r} by joining x_1 with y_1 and y_2 which is shown in Figure 3.6. It is clear that the eccentricity of every vertex is r and so $rad(G) = diam(G) = r$.

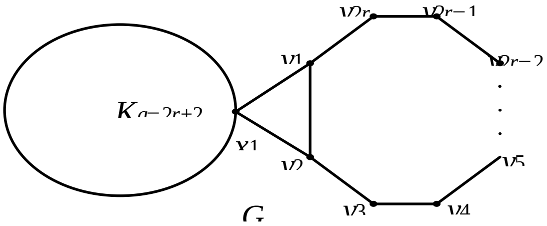

Figure 3.6

Here, $Ext(G) = \{x_2,x_3,...,x_{a-2r+2}\}$ and $Cut(G) = \{x_1\}$. Let $S = Ext(G) \cup Cut(G)$. Then $|S| = a-2r+2$. By Theorem 3.5 and Theorem 3.14, every connected geodetic global dominating set contains S. It is easily verified by Theorem 3.27, $S_1 = S \cup \{y_1,y_2,...,y_{2r-2}\}$ is a connected geodetic global dominating set of G and so $\overline{\gamma_{gc}}(G) \le |S_1| = a$. Now, we show that $\overline{\gamma_{gc}}(G) \ge a$. Suppose $\overline{\gamma_{gc}}(G) < a$. Let S'_1 be a connected geodetic global dominating set of G with $|S'_1| < a$. Then, S'_1 contains every vertex in K_{a-2r+2} and at most $2r - 3$ vertices in C_{2r}. Since, every vertex in C_{2r} to x_1 contains exactly two path that $S'_1 \cap \{y_1, y_2\} \ne \phi$. Then two subcases arise

Subcase (i) let $y_1 \in S'_1$ and $y_2 \notin S'_1$. Since S'_1 contains at most $2r - 3$ vertices in C_{2r}. By Theorem 3.27, $\{y_1, y_{2r}, y_{2r-1} ..., y_6, y_5\} \subseteq S'_1$. Thus, $y_4, y_3, y_2 \notin S'_1$. Clearly y_3 is not dominated by any vertex in S'_1 and so S'_1 is not a connected geodetic global dominating set in G, which is a contradiction.

Subcase (ii) let $y_1, y_2 \in S'_1$. Since S'_1 contains maximum $2r - 3$ vertices of C_{2r}, without loss of generality, let $\{y_1, y_2, ..., y_{2r-3}\} \subseteq S'_1$. Then $y_{2r}, y_{2r-1}, y_{2r-2} \notin S'_1$ and so no vertex in S'_1 dominates y_{2r-1}. Therefore, S'_1 is not a connected geodetic global dominating set of G again contradiction. Hence, $\overline{\gamma_{gc}}(G) = a$.

Case (ii) $r < d \le 2r$. Let $C_{2r} : x_1,x_2,...,x_{2r},x_1$ be a cycle of order of $2r$ and let $P_{d-r+1} : y_0,y_1,y_2,...,y_{d-r}$ be a path of order $d-r+1$. Let G be a graph obtained from P_{d-r+1} and C_{2r} by

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identifying y_0 in P_{d-r+1} and x_1 in C_{2r}. Also, adding $a-r-d+2$ new vertices $w_1,w_2,\ldots,w_{a-r-d+2}$ to G and join each $w_i(1 \le i \le a-r-d+2)$ to y_{2-r-1} in G and the graph G is shown in Figure 3.7.

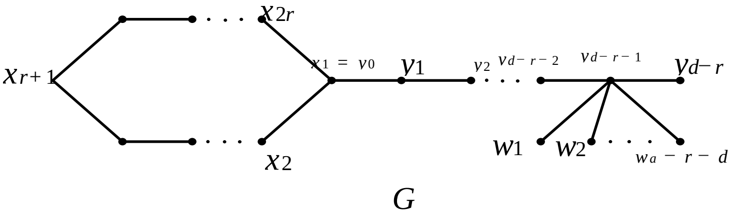

Figure 3.7

It is clear that $rad(G) = r$ and $diam(G) = d$. Also,
$Ext(G) = \{w_1,w_2,\ldots,w_{a-r-d+2}\}$ and $Cut(G) = \{x_1,y_1,y_2,\ldots,y_{d-r}\}$. Let $S = Ext(G) \cup Cut(G)$. By Theorem 3.5 and Theorem 3.14, S is contained in every connected geodetic global dominating set of G and $|S| = a - 2r + 3$. By Theorem 3.27, $S_1 = S \cup \{x_2,x_3,\ldots,x_{2r-2}\}$ is a connected geodetic global dominating set of G and so $\overline{\gamma_{gc}}(G) \le |S_1| = a$. As a similar argument as in case (i) we conclude that $\overline{\gamma_{gc}}(G) = a$. Hence the proof is complete.

3. CONCLUSIONS

In this paper we studied the properties of connected geodetic global domination number and discuss some named graphs. We can extended for some product of graphs.

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