Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

On Spherical Distributions

U. N. Bassey[1*], W. A. Latifu[2]

[1]Department of Mathematics, University of Ibadan, Ibadan, Nigeria*
[2]Faculty of Sciences, Mewar International University, Masaka, Nasarawa State, Nigeria

Corresponding Email: [1]unbassey@yahoo.com*

Received: 13 May 2022 **Accepted:** 26 July 2022 **Published:** 29 August 2022

***Abstract:** In this paper, we begin with the study of the hyperbolic spaces G/H where $G = U(p.q;\mathbb{F})$ and $H = U(1;\mathbb{F}) \times U(p-1,q;\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ denotes the set of real numbers, complex numbers and quaternions respectively. In the articles of J. Faraut [2] and M.T. Kosters and G. van Dijk [4], spherical distributions were derived following two different methods. The first method is to describe the behavior of spherical distributions making use of the Fourier transform of finite and infinite functions. The second is to express them as $M'S$ where $M' : \mathcal{H}' \to \mathcal{D}'(X)$ is a transpose map and S is a solution to the differential equation $LS \equiv a(t)S'' + b(t)S' = \lambda$ and making use of the hypergeometric functions. Now we show that spherical distributions T can be obtained through a particular distribution S on $\mathbb{R}$ by solving the equation $LS = \lambda S$. The technique of Methẻe's [6] is instrumental for the context.*

***Keywords:** Distribution, Spherical, Hyperbolic, Hypergeometric.*

1. INTRODUCTION

A distribution is a continuous linear functional on the space of test functions.
Let G be a Lie group and let H be a closed subgroup of G and let $X = G/H$ be the homogeneous space.
A differential operator on X is said to be invariant if it commutes with the action of G.
A distribution T on X is called spherical distribution if:
(i) T is eigendistribution of the Laplace operators ,i.e. $\square T=\lambda T$ with $\lambda \epsilon \mathbb{C}$.
(ii) T is H-invariant.

2. METHODOLOGY

1. The spaces $U(p,q;\mathbb{F})/U(1;\mathbb{F}) \times U(p-1,q;\mathbb{F})$

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

On $\mathbb{F}^n$,we consider the hermitian form:

$$[x,y]=\bar{y}_1 x_1+...+\bar{y}_p x_p-\bar{y}_{p+1} x_{p+1}-...-\bar{y}_{p+q} x_{p+q} \qquad (n=p+q)$$

and let $U(p,q;\mathbb{F})$ be the pseudo-unitary group preserving this norm.

Suppose $p\geq 1$. The group $G=U(p,q;\mathbb{F})$ acts on the projective space $\mathbb{P}_{n-1}=(\mathbb{F})$ and the stabilizer of the right coset generated by the vector $(1,0,...,0)$ is the group $H=U(1;\mathbb{F})\times U(p-1,q;\mathbb{F})$

The homogeneous space $X={G}/{H}$ is called hyperbolic space.

Now if X is an $n\times n$ matrix, we let $X^*=I_{pq}\bar{X}^t I_{pq}$ where $I_{pq}=\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$.

The Lie algebra $g=u(p,q;F)$ of G consist of $n\times n$ matrices X satisfying $X+X^*=0$. They are matrices of the form $\begin{pmatrix} X_1 & X_2 \\ \bar{X}_2^t & X_3 \end{pmatrix}$ where X_1 and X_3 are antihermitian and X_2 is arbitrary.

Let $J=\begin{pmatrix} 1 & & & & & 0 \\ & -1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & -1 \end{pmatrix}=I_{1,n-1}$ and we define the map $\theta: g\to g$ by $\theta(X)=J\times J$.

Then θ is an involutive automorphim of g.

The group of fixed points of θ is the Lie algebra h of H.

Write $L=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ Then $\theta(L)=-L$ and $a=RL$ is a maximal abelian subspace of $P=\{X\in g/\theta(X)=-X\}$.

Let m be the centralizer of L in h. Then $m=\left\{\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & u \end{pmatrix} \Big/ \begin{matrix} u\in F, u+\bar{u}=0 \\ v\in u(p-1,q-1;F) \end{matrix}\right\}$.

The nonzero eigenvalues of adL are $\alpha=1$ and $-\alpha$ if $F=R$ and $\alpha=1,2\alpha,-\alpha,-2\alpha$ if $F=\mathbb{C}$ or H.

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

For $\beta \in a^*$, let g^β be the simultaneous eigenspace for the elements $adX\,(X \in a)$ with eigenvalue $\beta(X)$. So $g^\beta = \{Y \in g : ad(X)Y = \beta(X)Y, \forall X \in a\}$. Then

$$g^\alpha = \left\{ X(z) = \begin{pmatrix} 0 & z^* & 0 \\ z & 0 & -z \\ 0 & z^* & 0 \end{pmatrix} \middle/ \begin{matrix} z \text{ is of type}(n-2,1) \\ z^* = -z^t I_{p-1,q-1} \end{matrix} \right\}$$

$$g^{2\alpha} = \left\{ Y(\omega) = \begin{pmatrix} \omega & 0 & -\omega \\ 0 & 0 & 0 \\ \omega & 0 & -\omega \end{pmatrix} \middle/ \begin{matrix} \omega \in F \\ \omega + \bar{\omega} = 0 \end{matrix} \right\}$$

$$g^\alpha = \theta(g^\alpha), g^{-2\alpha} = \theta(g^{-2\alpha})$$

and we give the following decomposition of g into eigenspaces of adL

$$g = a \oplus (\oplus g^\alpha)$$
$$= a \oplus \left(g^{-2\alpha} \oplus g^{-\alpha} \oplus m \oplus g^\alpha \oplus g^{2\alpha}\right)$$
$$\therefore g = g^{-2\alpha} \oplus g^{-\alpha} \oplus (m \oplus a) \oplus g^\alpha \oplus g^{2\alpha}.$$

Put $n = g^\alpha \oplus g^{2\alpha}$.

Then n is a nilpotent Lie subalgebra of g.

The subgroup A of G corresponding to the Lie subalgebra a is given by

$$A = \left\{ at = \exp(ta) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} : t \in R \right\}$$

Let $N = n(w,z) = \begin{pmatrix} 1 + w - \frac{1}{2}[z,z] & z^* & -w + \frac{1}{2}[z,z] \\ z & I & -z \\ w - \frac{1}{2}[z,z] & z^* & 1 - w + \frac{1}{2}[z,z] \end{pmatrix}$

where $w \in F$, $w + \bar{w} = 0$ and z is a matrix of type $(n-2,1)$ with coefficient in F, $z^* = -\bar{z}^t I_{p-1,q-1}$ and

$$z = \begin{pmatrix} z_2 \\ . \\ . \\ . \\ z_{n-1} \end{pmatrix}, z' = \begin{pmatrix} z'_2 \\ . \\ . \\ . \\ z'_{n-1} \end{pmatrix}$$

$$[z,z'] = \bar{z}'_2 z_2 + ... + \bar{z}'_p z_p - \bar{z}'_{p+1} z_{p+1} - ... - \bar{z}'_{p+q-1} z_{p+q-1}$$

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

The subgroup A normalizes N , that is, $a_t n(w,z)a_{-t} = n(e^{2t}w, e^t z)$.
Define the function $P_\circ$ on $\{y \in F^n / [y,y] = 0\}$ by

$$\tilde{P}_\circ(y) = \frac{|y_1 - y_n|}{\sqrt{[y,y]}}$$

The function $P_\circ$ is right-homogeneous of degree 0.
$\forall t \in F, t \neq 0, P_\circ(yt) = P_\circ(y)$ and it is written $\tilde{P}_\circ = P_\circ \circ \overline{w}$ where $P_\circ$ is a function defined on X and $\overline{w}: \mathbb{F}^n \to \mathbb{P}_{n-1}(\mathbb{F})$ is a canonical map.
Then Faraut ([2] p.390, proposition 4.2) proved the following analogue of the Iwasawa decomposition:

Proposition 1.1. Let $x \in X$ such that $P_\circ(x) \neq 0$. Then there exists an unique element a_t of A and a unique element $n(w,z)$ of N such that $x = n(w,z)a_t x^\circ$ where $x^\circ = eH$ .Also we have $t = \log P_\circ(x)$.

2. The cone $\mathbb{E} = G/MN$ and the Poison kernel

Here we present the Poison kernel and list some of its properties that we require in the sequel. We refer the reader to Faraut [2], Kosters and Van Dijk[4] for details.
Let Γ be the isotropic cone of the hermitian form $[y,y]$, that is, $\Gamma = \{y \in \mathbb{F}^n / [y,y] = 0, y \neq 0\}$
Let M be such that the Lie algebra of M is m .The homogeneous space $\mathbb{E} = G/MN$ can be identified as the quotient set $\Gamma/\sim$ where ~ denotes the equivalence relation on $\mathbb{F}^n$ defined by

$$y \sim y' \Leftrightarrow \exists\, u \in \mathbb{F}, |u| = 1, y' = yu.$$

Define for $x \in X$ and $\xi = gMN \in \mathbb{E}$ the map $P: X \times \mathbb{E} \to \mathbb{R}$ by $P(x,\xi) = P_\circ(g^{-1}x)$.
This map is the Poison kernel. It satisfies the following properties:
(i) $P(gx, g\xi) = P(x,\xi) \quad g \in G, x \in X, \xi \in \mathbb{E}$

(ii) $P(x,\xi) = \dfrac{|[y,\gamma]|}{\sqrt{[y,y]}}$

where x is the class of y and ξ is the class of γ
Define $P: \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ by $P(\xi,\xi') = |[\gamma,\gamma']|$ where γ and $\gamma' \in \Gamma$ such that ξ is the class of γ, ξ' is the class of γ'.

Then we obtain:
(iii) $P(g\xi^\circ,\xi) = 2\lim_{t\to\infty} e^{-t} P(ga_t\xi^\circ,\xi) \quad (g \in G, \xi \in E)$.

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

3. The representations π_s

The subgroup of G stabilizing the right coset generated by the vector $\gamma^\circ = (1,0,...,0,1)$, that is $\{g \in G / \ g\gamma^o \in \gamma^o F\}$ is a maximal parabolic subgroup of G equal to MAN [13].To every complex number we associate the character χ_S of the parabolic subgroup MAN defined by

$$\chi_S(ma_t n) = e^S$$

Define also the space $E_S(\mathbb{E})$ of functions on $\mathbb{E}$ by $E_S(\mathbb{E}) = \{f \epsilon C^\infty / f(ga_t\xi^o) = e^{(s-\rho)t} f(g\xi^o)\}$

where $g \in G$ and $\xi^o = eMN$.

Then the representation π_s of the group G induced by the character χ_s is defined as following: $(\pi_s(g)f)(\xi) = f(g\xi^{-1})$, $f \epsilon E_S(\mathbb{E}), \xi \epsilon \mathbb{E}, g \epsilon G$

As the homogeneous space $\mathbb{E} = G/MN$ is identified as the quotient space $\Gamma/\sim$, a function of $E_S(\mathbb{E})$ satisfies $f(\lambda\gamma) = |\lambda|^{s-\rho} f(\gamma), \forall\lambda \epsilon \mathbb{F}$, $\lambda \neq 0$.

Put $B = \{k\xi^o / k \in K\}$ where K is a subgroup of G equal to $U(p;\mathbb{F}) \times U(p;\mathbb{F})$.

Let db denote the normalized K-invariant measure on B.Then it implies that the non-degenerate bilinear form $<,>$ defined on $E_S(\mathbb{E}) \times E_{-S}(\mathbb{E})$ by $< f,h >= \int_B f(b)h(b)db$ is G-invariant [2] i.e.

$$\forall g \in G, < \pi_s(g)f, \pi_{-s}(g)h >=< f,h >.$$

Write $E_S'(\mathbb{E})$ for the topological dual of $E_{-S}(\mathbb{E})$ and let π'_s be the representation of G on $E_S'(\mathbb{E})$ such that $E_S(\mathbb{E})$ can be regarded as a subset of $E_S'(\mathbb{E})$.

Proposition 3.1: Let s be purely imaginary, $s \neq 0$. Then π_s is (topologically) irreducible.

Proof : Van dijk and Kosters [4] proved this theorem when $X = SL(n,R)/GL(n-1,R)$.

Now, let $f \to f^o$ be the projection of $C_c^\infty(G)$ onto E_s given by $f^o(g) = \int_{MAN} f(gma_t n)e^{(\rho-s)t} dmdtdn$.

dm and dn are here the respective Haar measures on M and N.

If $A: E_s \to E_s$ intertwines π_s then $(f,g) \to \int_B \overline{Af^o(b)} g^o(b)db$ $(f,g \in C_c^\infty(G))$ defines a continuous G-invariant sesquilnear form on $C_c^\infty(G)$. By the Schwartz kernel theorem [8] and the invariance of this form, there exists a distribution T on G such that

$$< T, \tilde{f} * g >= \int_B \overline{Af^o(b)} g^o db \qquad (f,g \in C_c^\infty(G))$$

and satisfying

$$T(ma_t ngm' a_x n') = e^{(s+\rho)t+(s-\rho)x} T(g).$$

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

Such distributions T are unique up to scalar multiples, provided $s \neq 0$.This is shown in [4] when applying the theory of Bruhat. The theorem is shown from the converse of Schur's lemma for unitary representations.

□

Define $P_1 : \mathbb{E} \to \mathbb{R}$ by $P_1(\xi) = p(x^o, \xi)$. If ξ is the class of a point γ of Γ, then $P_1(\xi) = |\gamma_1|$ where γ_1 is the first coordinate of γ.

For $\operatorname{Re} s > \rho$, the function F defined on $\mathbb{E}$ by $F(\xi) = P_1(\xi)^{s-\rho}$ defines an element of $E_s'(\mathbb{E})$ i.e. a continuous linear functional on $E_s(\mathbb{E})$:

$< F, f > = \int_B P_1(b)^{s-\rho} f(b)db$.

Also for all $h \in H$, $\pi_s(h)F = F$.

If $f \in E_{-s}(\mathbb{E})$, we define for $s \in \mathbb{C}$ $\operatorname{Re} s > \rho$

$$u_s(f) = \frac{1}{\Gamma\left(\frac{s-\rho+d}{2}\right)} \int_B P_1(b)^{s-\rho} f(b)db .$$

Faraut [2] showed the analytic continuation of the function $s \mapsto u_s(f)$.

We also remark that

$$\left(\pi_s'(g)u_s\right)f = \frac{1}{\Gamma\left(\frac{s-\rho+d}{2}\right)} \int_B P_1(g^- b)^{s-\rho} f(b)db$$

$$= \frac{1}{\Gamma\left(\frac{s-\rho+d}{2}\right)} \int_B P(gx^o, b)^{s-\rho} f(b)db .$$

Let $\phi \in \mathcal{D}(G)$

$$\left(\pi_s'(\phi)u_s\right)(f) = \int_G \left(\pi_s'(g)u_s\right)(f)\phi(g)dg$$

$$= \frac{1}{\Gamma\left(\frac{s-\rho+d}{2}\right)} \int_G \int_B P(gx^o, b)^{s-\rho} f(b)db\phi(g)dg .$$

Therefore $\pi_s'(\phi)u_s \in E_s(\mathbb{E})$, i.e. , $\left(\pi_s'(\phi)u_s\right)(\xi) = \frac{1}{\Gamma\left(\frac{s-\rho+d}{2}\right)} \int_G P(gx^o, \xi)^{s-\rho} \phi(g)dg$.

This shows that

$u_s \in E_s'(\mathbb{E})$ for all s and $\pi_s'(h) = u_s$ for all h.

If $\phi \in \mathcal{D}(G)$, we define $\zeta_s(\phi) = < \pi_s'(\phi)u_s, u_{-s} >$.

Then from ([2] p.396, proposition 5.4), the following proposition follows:

Proposition 3.2. ζ_s is a spherical distribution, that is, ζ_s is a H bi-invariant eigendistribution of □ for eigenvalue λ.

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

4. Intertwining operators

If $f \in C^{\infty}(B)$ and $s \in \mathbb{C}$ such that $\operatorname{Re} s > \rho$, we define

$W_s(f) = \int_B P(\xi^o, b)^{s-\rho} f(b)db$.

Faraut [2] shows that this function can be extended to a meromorphic function with simple or double poles in the set $(\rho - d - 2\mathbb{N}) \cup (-\mathbb{N})$ and that

$$W_s(1) = \frac{\Gamma\left(\frac{dp}{2}\right)\Gamma\left(\frac{dq}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \frac{\Gamma\left(\frac{s-\rho+d}{2}\right)\Gamma(s)}{\Gamma\left(\frac{s+\rho}{2}\right)\Gamma\left(\frac{s+\rho-dp+2}{2}\right)\Gamma\left(\frac{s+\rho-dq+2}{2}\right)}$$

Now let $f \in E_{-s}(\mathbb{E})$.
For $\operatorname{Re} s$ we get

$$\left(A_s(f)\right)(\xi) = \frac{1}{W_s(1)} \int_B P(\xi, b)^{s-\rho} f(b)db.$$

Then the following theorem [2, 4] follows:

Theorem 4. 1. (i) Suppose $s+\rho$ is not an integer. If $f \in E_{-s}(\mathbb{E})$ then $A_s f \in E_s(\mathbb{E})$ and the map $A_s: E_{-s}(\mathbb{E}) \to E_s(\mathbb{E})$ is continuous.
(ii) A_s intertwines the actions of G i.e. $A_s \circ \pi_s(g) = \pi_s(g) \circ A_s$.

The usual Laplace –Beltrami operators Δ_1 and Δ_2 of the unit spheres $S(\mathbb{F}^p)$ and $S(\mathbb{F}^q)$ act on the space $C^{\infty}(\Sigma)$ where $\Sigma = S(\mathbb{F}^p) \times S(\mathbb{F}^q)$.

$C^{\infty}(B)$ can be considered as a subspace of $C^{\infty}(\Sigma)$ i.e.

$C^{\infty}(B) \cong \left\{ f \in C^{\infty}(\Sigma) / \forall u \in F, |u| = 1, f(\sigma u) = f(\sigma) \right\}$.

Let us consider for two integers ℓ and $m \geq 0$ the subspace

$\mathcal{Y}_{\ell m} = \{ f \in C^{\infty}(B) / \Delta_1 f = -\ell(\ell + dp - 2) f, \Delta_2 f = -m(m + dq - 2) f \}$

Faraut [2] further shows that the $M \cap K$-invariant functions of $\mathcal{Y}_{\ell m}$ are proportional to the function w_{lm} defined by $w_{lm}(\sigma) = C_{\ell}^{\frac{(p-2)}{2}}(\sigma_1) C_m^{\frac{(q-2)}{2}}(\sigma_n)$ where C_{ℓ}^{λ} donates the $\ell - th$ Gegenbauer polynomial of index λ.

5. Determination of spherical distributions using Fourier transform

Definition 5.1. If $\phi \in \mathcal{D}(X)$ then the Fourier transform $\mathbb{E} \times \mathbb{C} \to \mathbb{C}$ is given by

$$\hat{\phi}(\xi, s) = \frac{1}{\Gamma\left(\frac{s-\rho+d}{2}\right)} \int_X P(x, \xi)^{s-\rho} \phi(x)dx \text{ for } \operatorname{Re} s > \rho - d.$$

Then we have from [2.4]:

(i) For fixed $\xi, \hat{\phi}(\xi, s)$ is an entire function of s

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

(ii) For fixed s, $\hat{\phi}(\bullet,s)\in E_S(\mathbb{E})$ that is ,it is a C^∞ – function of ξ satisfying $\forall t \in \mathbb{R}$, $\hat{\phi}(g_{a_t}\xi^o,s)=e^{(s-\rho)t}\hat{\phi}(g\xi^o,s)$.

(iii) The Fourier transform commutes with the action of G.

If we denote for $g\in G,\phi\in \mathcal{D}(X)$,the function $x\to\phi(g^{-1}x)$ by $\tau_g\phi$ then

$$(\tau_g\phi)^\wedge(\xi,s)=\pi_s(g)\hat{\phi}(\xi,s)$$

(iv) $(\Box\phi)^\wedge(\xi,s)=(s^2-\rho^2)\hat{\phi}(\xi,s)$.

For $\operatorname{Re} s<-\rho+d$, the spherical distributions on $X=G/H$ are

$$\zeta_s(\phi)=\frac{1}{\Gamma\left(\frac{-s-\rho+d}{2}\right)}\int_B\hat{\phi}(b,s)P_1(b)^{-s-\rho}db.$$

(a) Fourier transform of K – finite functions

Every element $x\in X$ can be written as $x=ka_tx^o$ with $t\geq 0,k\in K$. Since $U(p;\mathbb{F})$ is transitive on the unit sphere $S(\mathbb{F}^P)$ the point x depends uniquely on the class kM_o where $M_o=M\cap K$.

Meanwhile the pseudo-Laplacian $\Box$ on X is given by

$$\Box f=\frac{1}{A(t)}\frac{\partial}{\partial t}\left(A(t)\frac{\partial f}{\partial t}\right)-\frac{1}{\cosh^2 t}\Delta_1 f+\frac{1}{\sinh^2 t}\Delta_2 f$$

where $A(t)=\left(e^t+e^{-t}\right)^{dp-1}\left(e^t-e^{-t}\right)^{dq-1}$ and Δ_1 and Δ_2 are differential operators on B defined as above.

Therefore if $f\in\mathcal{Y}_{\ell m}$, then

$$\Box f=\frac{1}{(e^t+e^{-t})^{dp-1}(e^t+e^{-t})^{dq-1}}\frac{\partial}{\partial t}\left[(e^t+e^{-t})^{dp-1}(e^t+e^{-t})^{dq-1}\frac{\partial f}{\partial t}\right]+\frac{\ell(\ell+dp-2)f}{\cosh^2 t}$$
$$-\frac{m(m+dq-2)f}{\sinh^2 t}.$$

In case $m=\ell=p=q$ i.e. if $d=2$ then $A(t)=1$ and

$$\Box f=\frac{1}{1}\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right)+\frac{1(2-2)f}{\cosh^2 t}-\frac{1(2-2)f}{\sinh^2 t}$$

$$\therefore\Box f=\frac{\partial^2 f}{\partial t^2}$$

So we easily see that when $d=2$ i.e. for $X=\frac{U(1,1)}{U(1)\times U(1)}$, the pseudo-Laplacian is

$$\Box=\frac{\partial^2 f}{\partial t^2}.$$

Now consider a finite function $\phi\in\mathcal{D}(X)$ of type (ℓ,m) that is, of the form $\phi(x)=F(t)\Upsilon(b)$ where $\Upsilon\in\mathcal{Y}_{\ell m}$.

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

The function F can be written as $F(t)=t^m F_o(t)$ where F_o is an even function of $\mathcal{D}(\mathbb{R})$. Then we give the expression of Fourier transform for such functions.
(i) Suppose that $\mathbb{F} = \mathbb{R}$.
We consider $\Upsilon(b)$ as a function defined on
$\Sigma = S(\mathbb{R}^p) \times S(\mathbb{R}^q)$, $b = (\sigma', \sigma'')$,where $\sigma' \in S(\mathbb{R}^p)$ and $\sigma'' \in S(\mathbb{R}^q)$
and $\xi \mapsto \phi(\xi, s)$ is a function defined on the isotropic cone Γ .
$\xi = (\lambda\tau', \lambda\tau'')$,$\tau' = S(\mathbb{R}^p)$, $\tau'' = S(\mathbb{R}^q)$, $\lambda \in \mathbb{R}, \lambda \neq 0$
We obtain

$$\hat{\phi}(\xi,s) = \frac{4\pi^{\rho+1}}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} 2^{-2\rho} \frac{|\lambda|^{s-\rho}}{\Gamma\left(\frac{s-\rho+1}{2}\right)} \times \int_0^\infty \int_{S(\mathbb{R}^p)} \int_{S(\mathbb{R}^q)} |cosht\tau'.\sigma' - sinht\tau''.\sigma''|^{s-\rho} F(t)\Upsilon(\sigma',\sigma'')A(t)dtd\sigma' d\sigma''$$

Because of $K-$invariance, the restriction of $\hat{\phi}(\bullet,s)$ to B belongs to $\mathcal{Y}_{\ell m}$ and since $\mathcal{Y}_{\ell m}$ is irreducible with the action of K, we have $\hat{\phi}(\lambda\tau,s)=|\lambda|^{s-\rho}\tilde{F}(s)\Upsilon(\tau)$,
$\tilde{F}$ depending uniquely on (ℓ, m) and F.
To calculate $\tilde{F}(s)$,we consider the particular case

$$\Upsilon(\sigma)=\omega_{\ell m}(\sigma)=C_\ell^{\frac{(p-2)}{2}}(\sigma_1)C_m^{\frac{(q-2)}{2}}(\sigma_n).$$

We obtain $\tilde{F}(s)=\int_o^\infty \Phi_{\ell m}(t,s)F(t)A(t)dt$

$$\text{with } \Phi_{\ell m}(t,s)=\frac{4\pi^\rho}{\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{q-1}{2}\right)}2^{-2\rho}\frac{1}{\Gamma\left(\frac{s-\rho+1}{2}\right)}\left[C_\ell^{\frac{P-2}{2}}(1)C_\ell^{\frac{q-2}{2}}(1)\right]^{-1}$$

$$\times\int_{-1}^1\int_{-1}^1|\cosh\sigma_1-\sinh\sigma_n|^{s-\rho}C_\ell^{\frac{p-2}{2}}(\sigma_1)C_m^{\frac{q-2}{2}}(\sigma_n)\left(1-\sigma_1^2\right)^{\frac{p-2}{2}}d\sigma_1 d\sigma_n \ .$$

The relation $\square\phi^\wedge(\xi,s)=(s^2-\rho^2)\hat{\phi}(\xi,s)$ implies that the function $\Phi_{\ell m}(\bullet,s)$ is a solution of

$$\frac{1}{A(t)}\frac{d}{dt}\left(A(t)\frac{du}{dt}\right)-\left[\frac{\ell(\ell+p-2)}{\cosh^2 t}-\frac{m(m+q-2)}{\sinh^2 t}\right]u=(s^2-\rho^2)u\ .$$

This equation can be solved by substituting $\tanh^2 t$ as a new variable in the equation then we arrive at a hypergeometric differential equation. Solving the latter, we obtain (using the fact that $\Phi_{\ell m}(t,s)$ is regular for $t=0$) that $\Phi_{\ell m}(t,s)=\beta_{\ell m}(s)\Psi_{\ell m}(t,s)$
where

$$\Psi_{\ell m}(t,s)=(\tanh t)^m(\cosh t)^{s-\rho}{}_2F_1\left(\frac{\rho-s+m+\ell}{2},\frac{\rho-s-p+2+m-\ell}{2};m+\frac{q}{2};\tanh^2 t\right)$$

and the numbers
$\beta_{\ell m}(s)$ are calculated in [2]:

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

$$\beta_{\ell m}(s) = b_{\ell m}\frac{(s-\rho)(s-\rho-2)...(s-\rho-\ell+2)}{\Gamma\left(\frac{s-\rho+\ell-m+p}{2}\right)} \quad \text{where } b_{\ell m}\neq 0$$

We notice that $\Psi_{\ell m}(t,-s)=\Psi_{\ell m}(t,s)$.

Finally,

$$\hat{\phi}(\lambda t,s)=|\lambda|^{s-\rho}\Upsilon(\tau)\beta_{\ell m}(s)\int_o^{\infty}\Psi_{\ell m}(t,s)F(t)A(t)dt$$

(ii) When $\mathbb{F}=\mathbb{C}$ or $\mathbb{H}$, the results are the same.

In all cases $\hat{\phi}(b,s)=b_{\ell m}\Upsilon(b)\beta_{\ell m}(s)\int_o^{\infty}\Psi_{\ell m}(t,s)F(t)A(t)dt$,

where $\beta_{\ell m}(s)=b_{\ell m}\dfrac{(s-\rho)(s-\rho-2)...(s-\rho-m-\ell+2)}{\Gamma\left(\frac{s-\rho+\ell-m+dp}{2}\right)}$ and

$$\Psi_{\ell m}(t,s)=(\tanh t)^m(\cosh t)^{s-\rho}\,{}_2F_1\left(\frac{\rho-s+m+\ell}{2},\frac{\rho-s-dp+2+m-\ell}{2},m+\frac{dq}{2};\tanh^2 t\right)$$

(b) Expression of $\varsigma_s(\phi)$ when ϕ is $K-$finite.

Let $\phi(x)=F(t)\Upsilon(b) \quad (x=ka_t x^o, t\geq 0, b=kM_o)$ where $\Upsilon\in\mathcal{Y}_{\ell m}$

We make use of the expression

$$\varsigma_s(\phi)=\frac{1}{\Gamma\left(\frac{-s-\rho-d}{2}\right)}\int_B\phi(b,s)P_1(b)^{-s-\rho}db \quad \text{for } \operatorname{Re}s<-\rho+d$$

This implies that

$$\varsigma_s(\phi)=\beta_{\ell m}(s)\int_0^{\infty}\psi_{\ell m}(t,s)F(t)A(t)dt\times\frac{1}{\Gamma\left(\frac{-s-\rho+d}{2}\right)}\int_B\Upsilon(b)P_1(b)^{-s-\rho}db.$$

Let $\omega_{\ell m}$ be the function on $\sum$ defined by

$$\omega_{\ell m}(\sigma)=\int_{\mathrm{U}(1;\mathbb{F})}C_\ell^{\frac{p-2}{2}}(Re(\sigma_{1u}))C_m^{\frac{q-2}{2}}(Re(\sigma_{nu}))du.$$

Then $\omega_{\ell m}\in\mathcal{Y}_\ell^{dp}\otimes\mathcal{Y}_m^{dq}$ and satisfies

$\omega_{\ell m}(\sigma u)=\omega_{\ell m}(\sigma), \forall u\in F, |u|=1$.

And so, $\omega_{\ell m}\in\mathcal{Y}_{\ell m}$

In the distribution sense, we have

$$\frac{1}{\Gamma\left(\frac{s-\rho+d}{2}\right)}P_1(b)^{s-\rho}=\sum_{\ell\, even}c_\ell\beta_{\ell o}(s)\omega_{\ell o}(b)$$

where $c_\ell\neq 0$.

Finally, we obtain

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

$$\xi_s(\phi) = c_\ell \beta_{\ell m}(s)\beta_{\ell o}(-s)\int_0^\infty \Psi_{\ell m}(t,s)F(t)A(t)dt \times \int_B \Upsilon(b)\omega_{\ell o}(b)db$$

Theorem 5.2.

(i) $\xi_s = \xi_{-s}$ for all $s \in \mathbb{C}$.

(ii) If $p=1$, the distribution ξ_s vanishes for certain values of s but for all s ,the entire function $\Gamma\left(\frac{s-\rho+d}{2}\right)\Gamma\left(\frac{-s-\rho+d}{2}\right)\xi_s$ is non- zero.

(iii) If $\mathbb{F} = \mathbb{R}$, $p>1$ and q is odd, $\xi_s \neq 0$ for all s .

(iv) If $\mathbb{F} = \mathbb{R}$, $p>1$ and q is even or if $\mathbb{F} = \mathbb{C}$ *or* $\mathbb{H}$

$\forall s \neq \pm(\rho+2r), r = 0,1,2,...$ $\qquad \xi_s \neq 0$

$\forall s = \pm(\rho+2r), r = 0,1,2,...$ $\qquad \xi_s = 0$

Proof:

(i) The distributions ξ_s and ξ_{-s} are spherical distributions relative to the same eigenvalue $\lambda = s^2 - \rho^2$. From [2], they are proportional except for certain isolated values of s .When Φ is K- invariant, Kosters and Van Dijk [2] show that $\xi_s = \xi_{-s}$.

(ii) If $\mathbb{F} = \mathbb{R}$ and q is odd, for all s we can find a number ℓ such that $\beta_{\ell 0}(s)\beta_{\ell 0}(-s) \neq 0$ then $\xi_s = 0$.

(iii) If $\mathbb{F} = \mathbb{R}$ and q is even or if $\mathbb{F} = \mathbb{C}$ *or* $\mathbb{H}$ and if $s \neq \pm(\rho+2r)$ we can find a number ℓ such that $\beta_{\ell o}(s)\beta_{\ell o}(-s) \neq 0$.

However if $s = \pm(\rho+2r)$ for every pair (ℓ, m),

$\beta_{\ell m}(s)\beta_{lo} = 0$ showing that $\xi_s = 0$.

(c) Behavior of ξ_s at infinity

The following theorem [2] helps in discussing spherical distributions at infinity:

Theorem 5.3.

The Fourier transformation F_s satisfies $A_s \circ F_{-s} = \gamma(s) \circ F_s$ where A_s is the intertwining operator defined in $\oint 4$ and $\gamma(s) = \dfrac{\Gamma\left(\frac{s-\rho+dp}{2}\right)}{\Gamma\left(\frac{-s-\rho+dp}{2}\right)}$ is a meromorphic function.

Now, define the function $c(s)$ by $c(s) = \dfrac{2^{\rho-s}}{\Gamma\left(\frac{s-\rho+d}{2}\right)}\int_B P(\xi^o, b)^{s-\rho}db$ for $\mathrm{Re}\, s > \rho - d$.

Then

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

$$c(s)=\frac{\Gamma\left(\frac{dp}{2}\right)\Gamma\left(\frac{dq}{2}\right)}{\sqrt{\pi}}\times\frac{2^{\rho-s}\Gamma(s)}{\Gamma\left(\frac{s+\rho}{2}\right)\Gamma\left(\frac{s+dp-\rho}{2}\right)\Gamma\left(\frac{s+dq-\rho}{2}\right)}$$

Proposition 5.4.
Let $\phi \in \mathcal{D}(X)$
Let $\phi_t(x)=\phi(a_t x)$.
For $\operatorname{Re} s>\rho-d$ we have

$$\lim_{t\to\infty} e^{-(s-\rho)t}\zeta_s(\phi_t)=c(s)\gamma(s)\hat{\phi}(\xi^o,s).$$

Proof:
We have $\zeta_s(\phi_t)=\frac{1}{\Gamma\left(\frac{s-\rho+d}{2}\right)}\int_B \hat{\phi}(b,s)P_1(a_{-t}b)^{s-\rho}\,db$

And then

$$\lim_{t\to\infty} e^{-(s-\rho)t}\zeta_s(\phi_t)=\frac{2^{\rho-s}}{\Gamma\left(\frac{s-\rho+d}{2}\right)}\int_B \hat{\phi}(b,-s)P_1(\xi^o,b)^{s-\rho}\,db.$$

By Theorem 5.3, $\lim_{t\to\infty} e^{-(s-\rho)t}\zeta_s(\phi_t)=c(s)\gamma(s)\hat{\phi}(\xi^o,s)$.

6. Determination of spherical distributions using the hypergeometric function.

In this section, we evaluate for each λ the space $\mathcal{D}'_{\lambda,H}(X)$ of H-invariant eigendistributions of □ for eigenvalue λ.
The basic ideas are identical to those in [2,4].
We define $Q: X\to\mathbb{R}$ satisfying the following properties:

(a) Q is $H-$invariant.
(b) Q is real analytic.
(c) x^o is a non-degenerate critical point for Q. The Hessian of Q in this point has signature $(dq, d(p-1))$.
(d) If $p=1, Q(x)\geq 1$.
(e) If $p>1, Q\geq 0$. If $Q(x)=0$, then x is a degenerate critical point and the manifold $\{Q(x)=0\}$ is a non-degenerate critical manifold.

The Hessian of Q in this point has signature $(d,0)$.
Let F be a continuous function on $\mathbb{R}$. We consider the distribution T defined on X by

$$T(f)=\int F[Q(x)]f(x)dx,\ f\in\mathcal{D}(x)$$

T is H-invariant.
If t is not a critical point of Q we can define the mean value Mf on $\{Q(x)=t\}$ by

$$Mf(t)=\int_X f(x)\delta(Q(x)-t)dx.$$

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

Then for every continuous function F on $\mathbb{R}$,

$$\int_X F[Q(x)]f(x)dx = \int_{\mathbb{R}} F(t)Mf(t)dt$$

The function Mf has singularities in the critical points of Q that are $t=1$ and 0.

(a) If $p=1$, then $\sup Mf \subset [1,\infty)$ and

$Mf(t) = (t-1)^{\frac{dq}{2}-1}\phi_1(t)$ where $\phi_1 \in \mathcal{D}([1,\infty))$

(b) If $p>1$, then $\sup Mf \subset [0,\infty)$ and

$Mf(t) = t^{\frac{d}{2}-1}[\phi_o(t)+\gamma(t)\phi_1(t)]$, where ϕ_o and ϕ_1 are functions of $\mathcal{D}([o,\infty))$ and γ depends on d, p and q as following:

d	p	q	$\gamma(t)$
1	odd	odd	$\Upsilon(t-1)(t-1)^{\mu}$
	even	odd	$(t-1)^{\mu} Log\lvert t-1\rvert$
	even or odd	even	$\Upsilon(1-t)(1-t)^{\mu}$
2 or 4	-	-	$\Upsilon(1-t)(1-t)^{\mu}$

where $\mu = \dfrac{d}{2(d+p-1)} - 1$ and Υ denotes the Heaviside function $\Upsilon(t)=1$ if $t \geq 0$ and $\Upsilon(t)=0$ if $t<0$.

Now, let $\mathcal{H}=\mathcal{H}(d,p,q)$ be the space of functions ϕ defined on $[0,\infty)$ of the following form:

(a) If $p=1$ $\phi(t)=(t-1)^{\frac{dq}{2}-1}\phi_1(t)$ where $\phi_1 \in \mathcal{D}([1,\infty))$.

(b) If $p>1$ $\phi(t)=t^{\frac{d}{2}-1}[\phi_o(t)+\gamma(t)\phi_1(t)]$ where ϕ_o and ϕ_1 are functions of $\mathcal{D}([0,\infty))$

We easily see that for $p \geq 1$, the functions ϕ of $\mathcal{H}$ are of the form $\phi(t)=\phi_o^{'}(t)+\gamma(t)\phi_1^{'}(t)$ where $\phi_o^{'}$ and $\phi_1^{'}$ are functions of $\mathcal{D}(\mathbb{R})$.

Define a function $\mathcal{X}: \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ by $\mathcal{X}(f) = 1$ in the 0-neighborhood.

On the space $\mathcal{H}_\tau = \{\emptyset \in \mathcal{H} \ / \ supp\emptyset \subset [-\tau,\tau]\}$

We consider for every integer n and every integer $m < n + \dfrac{\mu}{2}$ the seminorm

$p_1(t) = \sup_t \left| \left(\dfrac{d}{dt}\right)^m \left[\phi(t) - \chi(t)\gamma(t)\sum_{k=0}^{n} B_k t^k\right]\right|$ and for every integer $k \geq 0$ the seminorm

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

$p_2(t) = |B_k(\phi)|$.

The seminorms define on $\mathcal{H}_\tau$ a Frechet topology. $\mathcal{H}$ is the inductive limit of the spaces $\mathcal{H}_\tau$ .It consists of all ϕ with compact supports belonging to C^∞ for $t \neq 0$ and satisfying the following fact:

There exists numbers $B_k = B_k(\phi)$ such that for every n, $\phi(t) - \gamma(t)\sum_{k=0}^{n} B_k t^k \in C^n$ whenever $n + \frac{\mu}{2} > m$.Therefore $\mathcal{H}$ is equipped with the inductive limit topology.

Then we have:

Theorem 6.1

(a) $M: \mathcal{D}(X) \to \mathcal{H}$ is continuous and surjective.

(b)The image $M'(\mathcal{H}')$ of the transpose map M': $\mathcal{H}' \to \mathcal{D}'(X)$ between the dual topological vector spaces consists of all H -invariant distributions on X .

(c)$\Box \circ M' = M' \circ L$.

Then follows the following important result [2] on the space of spherical distributions:

Theorem 6.2.

Let $\mathcal{D}'_{\lambda,H}$ be the space of H -invariant distribution T on X and that are solutions of $\Box T = \lambda T$.

(a) If $p = 1$, for all $\lambda \in \mathbb{C}$, dim $\mathcal{D}'_{\lambda,H}(X) = 1$.

(b) Let $p > 1$.

(i) If $d = 1$ and q is odd and $\lambda \in \mathbb{C}$ then $dim\mathcal{D}'_{\lambda,H}(X) = 1.$

(ii) If $d = 1$ and q is even or if d=2 or 4 for every $\lambda \neq 2r(2r + d(p+q) - 2), r = 0,1,2,...,$ then $dim\mathcal{D}'_{\lambda,H}(X) = 1$

and for every $\lambda = 2r(2r + d(p+q) - 2), r = 0,1,2,..., then$

$dim\mathcal{D}'_{\lambda,H}(X) = 2.$

Now we express spherical distributions in terms of the hypergeometric function.

Consider on the interval $[0,\infty)$ the differential equation

$$Ly - \lambda y = a(t)y'' + b(t)y' - \lambda y = 0 \qquad (E)$$

with

$$a(t) = 4t(t-1)$$

$$b(t) = 4\left[\left(\frac{d}{2} + \mu + 1\right)t - \frac{d}{2}\right].$$

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

The solutions of (E) can be expressed in terms of the hypergeometric function. In [1], it is established that the unique solution of (E) defined on the complex plane $(-\infty, 0]$ *is*

$$\Phi(t,\lambda,\mu)={}_2F_1\left(\frac{\frac{d}{2}+\mu+s}{2},\frac{\frac{d}{2}+\mu-s}{2};1+\mu;1-t\right)$$

where $\mu = s^2-\left(\mu+\frac{d}{2}\right)^2$ is not a negative integer.

For $|t-1|<1$ i.e. $t\in(0,2)$,

$$\Phi(t,\lambda,\mu)=\sum_{k=0}^{\infty}a_k(\lambda,\mu)\frac{(1-t)^k}{k!}$$

with

$$a_k(\lambda,\mu)=\frac{\left(\frac{\frac{d}{2}+\mu+s}{2}\right)_k\left(\frac{\frac{d}{2}+\mu-s}{2}\right)_k}{(1+\mu)_k}$$

where

$$(\alpha)_k=\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}=\alpha(\alpha+1)...(\alpha+k-1).$$

On the other hand, the equation (E) has an analytic solution in O which is

$$F(t,\lambda,\mu)={}_2F_1\left(\frac{\frac{d}{2}+\mu+s}{2},\frac{\frac{d}{2}+\mu-s}{2};\frac{d}{2}\right)$$

defined on $[1,\infty)$

(a) If μ is not an integer, the function $W(t,\lambda,\mu)=\frac{\Phi(t,\lambda,\mu)}{(1-t)^{\mu}}$ is a solution of (E) and we have

$[1; pp.108-109]$:

$$F(t,\lambda,\mu)=A(\lambda,-\mu)\Gamma(-\mu)\Phi(t,\lambda,\mu)+A(\lambda,\mu)\Gamma(\mu)\frac{\Phi(t,\lambda,-\mu)}{(1-t)^{\mu}}$$

with

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

$$A(\lambda,\mu)=\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{\frac{d}{2}+\mu+s}{2}\right)\Gamma\left(\frac{\frac{d}{2}+\mu-s}{2}\right)}$$

(b) If μ is an integer ≥ 0, then we have if $t\in(0,2)$ then

$$F(t,\lambda,\mu)=A(\lambda,\mu)\Gamma(\mu)(1-t)^{-\mu}\sum_{k=0}^{\mu-1}a_k(\lambda,-\mu)\frac{(1-t)^k}{k!}+$$
$$\frac{(-1)^{\mu+1}}{\mu!}A(\lambda,-\mu)Log(1-t)\Phi(t,\lambda,\mu)+\sum_{k=0}^{\infty}b_k(\lambda,\mu)\frac{(1-t)^k}{k!}$$

with

$$b_k(\lambda,\mu)=\lim_{\nu\to\mu}\left[A(\lambda,-\nu)\Gamma(-\nu)a_k(\lambda,\nu)+A(\lambda,\nu)\Gamma(\nu)a_{k+\mu}(\lambda,-\nu)\right].$$

Now let S be an element of the space $\mathcal{H}'(d,p,q)$ which is solution of (E) where $\mu=\frac{d}{2}(p+q-1)$.

On each of these intervals $(0,1)$ and $(1,\infty)$, S is an ordinary solution of (E). Also this solution must be regular at 0 showing that its restriction to $(1,0)$ is proportional to $F(\bullet,\lambda,\mu)$. The regularity for O helps in defining the distributions S_λ and T_λ

$$S_\lambda(\phi)=\lim_{\varepsilon\to 0}\int_0^\infty\frac{1}{2}\left[F(t+i\varepsilon,\lambda,\mu)+F(t-i\varepsilon,\lambda,\mu)\right]\phi(t)dt$$

$$T_\lambda(\phi)=\lim_{\varepsilon\to 0}\int_0^\infty\frac{1}{2i}\left[F(t+i\varepsilon,\lambda,\mu)-F(t-i\varepsilon,\lambda,\mu)\right]\phi(t)dt.$$

We remark that if $\lambda=\lambda_r=2r(2r+d+2\mu)$, then $T_\lambda=0$. We now show that every $\emptyset\,\epsilon\mathcal{H}(d,p,q)$ has a unique expansion in the neighborhood of 1.

Let m be a fixed integer ≥ 1 and denote by $P_\upsilon(t)$ polynomials of degree $\leq\upsilon$ divisible by t^m. In particular $P_\upsilon=0$ unless $\upsilon\geq m$.

We have from [9] that

$\gamma(t)P_\upsilon\in C^\upsilon$ if and only if $P_\upsilon=0$. (1)

In the case of $\mathcal{H}$, $\gamma(1)P_\upsilon(t)\in C^\infty$.

We shall consider functions of class C^υ outside the origin

$\phi-\gamma P_\upsilon\in C^\upsilon$ (2)

for at least one polynomial P_υ.

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

It follows from (1) that P_υ is uniquely determined by ϕ. It is convenient to write $P_\upsilon = \sum_0^\upsilon B_k(\phi)t^k$ where the coefficient $B_k(\phi)$ are linear function of ϕ and $B_k(\phi)=0$ if $\phi \in C^\upsilon$ at the origin. Expanding (2) in Taylor series around $t=0$,

$$\phi(t)-\gamma(t)P_\upsilon(t)=\sum_0^\upsilon A_k(\phi)t^k+o(t^\upsilon).$$

We obtain another set B_j of functionals ϕ with supports at the origin.

Then,

$$\phi(t)=\sum_0^\upsilon A_k(\phi)t^k+\gamma(t)\sum B_k(\phi)t^k+o(t^\upsilon).$$

As $\upsilon \to \infty$

$$\phi(t)=\sum_0^\infty A_k(\phi)t^k+\gamma(t)\sum_0^\infty B_k(\phi)t^k.$$

Therefore in the neighbourhood of 1, every function ϕ with the property (2) has a unique expansion of the form

$$\phi(t)=\sum_0^\infty (1-t)^k A_k(\phi)+\gamma(t)\sum_0^\infty (1-t)^k B_k(\phi).$$

From [2: Proposition A.5, Proposition 3.3] and the ordinary solutions of (E), the following proposition follows.

Proposition 6.3 .

Let $p>1$.

(a) $\mathbb{F}=\mathbb{R}\,(d,1)$, p odd, q odd.

$LS_\lambda-\lambda s_\lambda=0$

$LT_\lambda-\lambda T_\lambda=4A(\lambda,\mu)\Gamma(\mu+1)\sin\pi\mu B_o$

(b) $\mathbb{F}=\mathbb{R}\quad(d=1)$, p even, q odd.

$LS_\lambda-\lambda s_\lambda=0$

$LT_\lambda-\lambda T_\lambda=(-1)^{\mu+1}4\pi A(\lambda,\mu)\Gamma(\mu+1)B_o$

(c) $\mathbb{F}=\mathbb{R}\ (d=1)$, p odd, q even

$LS_\lambda-\lambda s_\lambda=-4A(\lambda,\mu)\Gamma(\mu+1)B_o$

$LT_\lambda-\lambda T_\lambda=0$

(d) $\mathbb{F}=\mathbb{R}\ (d=1)$, p and q even or $F=\mathbb{C}$ or $H(d=2or4)$

$LS_\lambda-\lambda S_\lambda=-4A(\lambda,\mu)\Gamma(\mu+1)B_o\cos\pi\mu B_o$

$LT_\lambda-\lambda T_\lambda=0$.

We denote by $\mathcal{H}'_\lambda$ the space of a solution of (E) in the space $\mathcal{H}'=\mathcal{H}'(d,p,q)$ .We can now describe the generators of $\mathcal{H}'_\lambda$ and then the generators of $\mathcal{D}'_{\lambda,H}(X)$.

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

Theorem 6.4.

(a) If $p=1$, then $\mathcal{H}'_\lambda$ is generated by the linear functional $\phi \mapsto \int_1^\infty \Phi(t,\lambda,\mu)\phi(t)dt$.

(b) Let $p>1$

(i) If $\mathbb{F}=\mathbb{R}$, q odd, then the space $\mathcal{H}'_\lambda$ is generated by S_λ and $\mathcal{D}'_{\lambda,H}(X)$ is generated by $M^{'}S_\lambda$

(ii) If $\mathbb{F}=\mathbb{R}, q$ even or if $\mathbb{F}=\mathbb{C}$ or $\mathbb{H}$. For $\lambda \neq \lambda r = 2r(2r+d(p+q)-2)$, $r=0,1,2,...,$ $\mathcal{H}'_\lambda$ is generated by T_λ and $\mathcal{D}'_{\lambda,H}(X)$ is generated by $M^{'}T$.

For, $\lambda = \lambda r = 2r(2r+d(p+q)-2)$, $r=0,1,2,...,$ $\mathcal{H}'_\lambda$ is generated by $S_{\lambda r}$ and $T^{'}_{\lambda r}$ defined by $T_{\lambda r} = \lim_{\lambda\to\lambda_r} \frac{T_\lambda}{A(\lambda,\mu)}$ and $\mathcal{D}'_{\lambda,H}(X)$ is generated by $M^{'}S_{\lambda_r}$ and $M^{'}T^{'}_{\lambda_r}$.

Proof:

(a) If $p=1$,every element of $\mathcal{H}'_\lambda$ is proportional to the distribution S_1 defined by

$$S_1(\phi)=\int_1^\infty \Phi(t,\lambda,\mu)\phi(t)dt.$$

Therefore $\mathcal{H}'_\lambda$ is generated by the functional $\phi \mapsto \int_1^\infty \Phi(t,\lambda,\mu)\phi(t)dt$.

(b) Let $p>1$

(i) If $\mathbb{F}=\mathbb{R}$, q odd.

In the preceding proposition we have $LS_\lambda - \lambda S_\lambda = 0$ and $LS_\lambda - \lambda S_\lambda = 0$ for the case q odd, showing then that $\mathcal{H}'_\lambda$ is generated by S_λ.

Meanwhile in [2: Proposition A.4], the elements of $\mathcal{D}'_{\lambda,H}$ are described to be of the form $T=M^{'}S$ where S is an element of $\mathcal{H}'$solution of the differential equation (E) i.e. $LS \equiv a(t)S^{''} + b(t)S^{'} = \lambda S$ .This implies that $\mathcal{D}'_{\lambda,H}(X)$ is generated by $M^{'}S_\lambda$.

(ii)The method to prove this case is identical to the previous one.

7. Determination of spherical distributions using Methée's work

A Lorentz rotation of the space $\mathbb{R}^n$ is any linear homogeneous transformation $x_i' = \sum_{k=1}^{n} a_{ik}x_k$ $(i=1,...,n)$ that leaves invariant the quadratic form $t = x_n^2 - \sum_{i=1}^{n-1} x_i^2$ and where the determinant a_{ik} is equal to 1.

Let $v = x_n$ and $w = \left(\sum_{i=1}^{n-1} x_i^2\right)^{\frac{1}{2}}$ so that $t = v^2 - w^2$.

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

The domain defined in $\mathbb{R}^n$ by $v > 0$ and $t > 0$ is called interior of the future cone denoted by Ω_1 .The interior of the past cone defined by $v < 0$ and $t > 0$ is denoted by Ω_3 and exterior domain of these two cones defined by $t < 0$ is denoted by Ω_2.

Every Lorentz rotation that leaves Ω_2 invariant is called a proper rotation. But if Ω_1 and Ω_3 are either invariant or permutable with the action of the Lorentz rotation then the rotation is said to be improper.

We call invariant distribution every distribution T satisfying $\lambda T = T$ for every proper rotation λ. If T is invariant and if λ_1 and λ_2 are improper rotations then $\lambda_1 T = \lambda_2 T$ because $\lambda_2 \lambda_1^{-1}$ is a proper rotation.

Define $f : \mathbb{R}^n \to \mathbb{R}$

$x = (x_1, ..., x_{n-1}, v) \mapsto fx$

The point fx has x-coordinate $t = v^2 - \sum_{i=1}^{n-1} x_i^2$.

We denote by f_+ the restriction of f to the exterior of the past cone $C\overline{\Omega}_3$ and by f_- its restriction to the future cone $C\overline{\Omega}_1$.

The inverse images $f^{-1}t_o$, $f_+^{-1}t_o$ and $f_-^{-1}t_o$ of a point of $\mathbb{R}$ of negative x-coordinate $t_o < 0$ are all identical to an hyperboloid [6]. For every t_o, each of the sets $f_+^{-1}t_o$ and $f_-^{-1}t_o$ is invariant under the group G of proper rotations and transitive on it.

Thereafter, every invariant function $\phi(x)$ defined on $\mathbb{R}^n - O$ (O is the origin) is constant on each of these sets.

Let $\phi^+(t)$ be its value on $f_+^{-1}t$ and $\phi^-(t)$ its value on $f_-^{-1}t$.

We have

$$\phi = f_+^* \phi^+ \text{ in } C\overline{\Omega}_3, \ \phi = f_-^* \phi^- \text{ in } C\overline{\Omega}_1 \quad (3)$$

Then [6]:

$$\phi^+(t) = \phi^-(t) \ \text{ for } t > 0 \quad (4)$$

Conversely, to every pair (ϕ^+, ϕ^-) of functions on $\mathbb{R}$ and satisfying (4) corresponds an invariant function ϕ in $\mathbb{R}^n - O$ defined by (3).

Meanwhile, De Rham [9] stated that if O is the only critical point of the map f then to every distribution S in $\mathbb{R}'$ corresponds a distribution $f^* S$ defined in $\mathbb{R}^n - O$ but in general in $\mathbb{R}^n$ .However, if suppS then the centre O does not adhere to the support of $f^* S$ in $\mathbb{R}^n - O$ and the distribution $f^* S$ is thus defined in $\mathbb{R}^n$ considering the fact $f^* S = 0$ in the O-neighborhood.

Then to a Dirac distribution S_ε in R (representing a mass $+1$ at the point $\xi = \varepsilon$) corresponds in R^n a distribution $H_\varepsilon = f^* \delta_\varepsilon$ which is defined in R for every integer $\varepsilon \neq 0$ and for which the support is the hyperboloid $t = \varepsilon$ .m

To the kth derivative $S_\varepsilon^{(k)}$ of S_ε corresponds similarly a distribution $H_\varepsilon^\kappa = f^* S_\varepsilon^{(\kappa)}$ with support the same hyperboloid. Let $\Upsilon(\xi)$ be the Heaviside function $\Upsilon(\xi) = 1$ for $\xi > 0$ and $\Upsilon(\xi) = 0$ for

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

$\xi < 0$.To the distribution $\Upsilon_\varepsilon = \Upsilon(\xi - \varepsilon)$ corresponds the distribution $f^* \Upsilon_\varepsilon = \Upsilon(t - \varepsilon)$ equal to 1 in the region $t > \varepsilon$ and equal to O in the region $t < \varepsilon$.These formulae

$$\frac{d\Upsilon_\varepsilon}{d\varepsilon} = -S_\varepsilon \text{ and } \frac{dS_\varepsilon^{(k)}}{d\varepsilon} = -S_\varepsilon^{(k+1)} \quad (5)$$

imply

$$\frac{d}{d\varepsilon}\Upsilon(t-\varepsilon) = -H_\varepsilon, \quad \frac{d}{d\varepsilon}H_\varepsilon^k = -H_\varepsilon^{k+1} \quad (6)$$

$(\Upsilon(t-\varepsilon)$ is infinitely differentiable for $\varepsilon \neq 0$ but for $\varepsilon = 0)$.

However Methėe [6] stated also that $f_+^{-1}\varepsilon$ is the upper plane $(t > 0)$ of the hyperboloid $t = \varepsilon(\varepsilon > 0)$ and that $f_-^{-1}\varepsilon$ the lower plane $(t < 0)$ of the same hyperboloid.

Definition 7.1

Given a function $g(\varepsilon)$ defined for $\varepsilon > 0$, we see that there exists a unique linear combination $I(\varepsilon)$ of functions of the form $\varepsilon^\lambda \log_\varepsilon^\mu$, where μ is an integer ≥ 0 and λ is a number of which the real part is ≤ 0. Also $\lambda = 0$ is excluded if $\mu = 0$ such that $g(\varepsilon) - I(\varepsilon)$ tends towards a finite limit when $\varepsilon \to +0$.

This linear combination $I(\varepsilon)$ if it exist is called infinite part of $g(\varepsilon)$ for $\varepsilon \to +0$ and the limit in question is called finite part (originally called "partie finie") of $g(\varepsilon)$ for $\varepsilon \to +\infty$ and

$$\underset{\varepsilon \to +0}{Pf}\ g(\varepsilon) = \lim_{\varepsilon \to +0}(g(\varepsilon) - I(\varepsilon)).$$

If there is no linear combination $I(\varepsilon)$, say that $\underset{\varepsilon \to +0}{Pf}_{g(\varepsilon)}$ is meaningless.

If $g(\varepsilon)$ is defined for $\varepsilon < 0$, we similarly define $\underset{\varepsilon \to -0}{Pf}_{g(\varepsilon)}$ for $\varepsilon \to 0$.

De Rham [9] also stated that $\underset{\varepsilon \to +0}{Pf}\ \Upsilon(t-\varepsilon)t^h$ always exists and then represents a distribution well-defined in $\mathbb{R}^n$.

Now to obtain spherical distributions, we need to solve $\Box T = \lambda T \Rightarrow (\Box - \lambda)T = 0$.Let $k = -\lambda$ so we have $(\Box + k)T = 0$.The pair of distributions in R associated with every invariant distribution of the equation $(\Box + k)T = 0$ consists of distributions satisfying $(L + \kappa)F = 0$ or $(L_n + \kappa)F = 0$, where $L_n = 4t\frac{d^2}{dt^2} + 2n\frac{d}{dt}$.

But,

$$L_n \frac{d}{dt} = \frac{d}{dt}L_{n-2} \text{ and for } n \text{ even } (L_n + k)\left(\frac{d}{dt}\right)^{\frac{n-1}{2}} = \left(\frac{d}{dt}\right)^{\frac{n-1}{2}}(L_1 + k) \quad (7)$$

If V satisfies $(L_1 + k)V = 0$,then $F = \left(\frac{d}{dt}\right)^{\frac{n-1}{2}} V$ satisfies $(L_n + k)F = 0$.

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

Conversely if F satisfies this last equality and if F is the derivative of order $\frac{1}{2}(n-1)$ of certain V_O, we have $\left(\frac{d}{dt}\right)^{\frac{n-1}{2}}(L_1+k)V_O=0$, then $(L_1+k)V_O$ is a polynomial of degree $<\frac{1}{2}(n-1)$.

If $k\neq 0$ then there exists a polynomial P of degree $<\frac{1}{2}(n-1)$ such that $(L_1+k)p=(L_1+k)V_O$.

Thus $V=V_O-p$ satisfies $(L_1+k)V=0$ and we have $F=\left(\frac{d}{dt}\right)^{\frac{n-1}{2}}V$.

With a similar reasoning in the case where n is even, we arrive at the following result.

If n is odd (or even) and $k\neq 0$, the solution F of the equation $(L_n+k)F=0$ are the derivatives of order $\frac{1}{2}(n-1)\left(or\ \frac{1}{2}(n-2)\right)$ of the solutions V of the equation $(L_1+k)V=0\left(or (L_2+k)V=0\right)$.

First, we let n be odd.

Consider $(L_1+\kappa)V=0$. The usual solutions of this equation are [6].

$g_1(t)=\cos\sqrt{kt}$ and $|t|^{\frac{1}{2}}g_2(t)$ where $g_2(t)=\frac{\sin\sqrt{kt}}{\sqrt{kt}}$ (which are both entire functions of t).

From [6:∮ 9] we see that

$$V=ag_1(t)+bPf\Upsilon_\varepsilon g_1(t)+cPft^{\frac{1}{2}}g_2(t)+dPf(1-\Upsilon_{-\varepsilon})|t|^{\frac{1}{2}}g_2(t)+\sum e_m\delta^m$$

where $\sum e_m\delta^m$ is a distribution with support the origin of $\mathbb{R}$.

It is clear that the distribution $g_1(u)$ satisfies $(L_1+\kappa)V=0$

From [6:(8.1), (8.5) and (10,2)], we get for $n=1$

$$(L_1+k)\Upsilon_\varepsilon g_1(t)=\left(-2g_1(\varepsilon)+4\varepsilon g_1'(\varepsilon)\right)\delta_\varepsilon+4\varepsilon g_1(\varepsilon)\delta_\varepsilon'$$

$$(L_1+k)\Upsilon_\varepsilon t^{\frac{1}{2}}g_2(t)=4\varepsilon^{\frac{3}{2}}g_2'(\varepsilon)\delta_\varepsilon+4\varepsilon^{\frac{3}{2}}g_2(\varepsilon)\delta_\varepsilon'$$

$$(L_1+k)(1-\Upsilon_{-\varepsilon})|t|^{\frac{1}{2}}g_2(t)=4\varepsilon^{\frac{3}{2}}g_2'(-\varepsilon)\delta_{-\varepsilon}+4\varepsilon^{\frac{3}{2}}g_2(-\varepsilon)\delta_{-\varepsilon}' \qquad (8)$$

whence

$$(L_1+k)Pf\Upsilon_\varepsilon g_1(t)=-2\delta$$

$$(L_1+k)Pf\Upsilon_\varepsilon t^{\frac{1}{2}}g_2(t)=(L_1+k)Pf(1-\Upsilon_{-\varepsilon})|t|^{\frac{1}{2}}g_2(t)=0.$$

Then from [6: (5.2)] follows

$$(L_1+k)\delta^{(m)}=-2(2m+3)\delta^{(m+1)}+k\delta^{(m)}.$$

Therefore the general solution V of $(L_1+k)V=0$ is an arbitrary linear combination of the following three distributions

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

$$Pf\Upsilon_{\varepsilon}t^{\frac{1}{2}}g_2(t)\,,\,Pf\,(1-\Upsilon_{-\varepsilon})|t|^{\frac{1}{2}}g_2(t)\,,\;g_1(t) \tag{9}$$

From the remark above, the general solution of $(L_n+k)F=0$ for n odd is an arbitrary linear combination of the derivatives of order $\frac{1}{2}(n-1)$ of these three distributions (9).

And Pf and $\frac{d}{dt}$ being permutable, we have [6].

$$F_1=Pf\left(\frac{d}{dt}\right)^{\frac{n-1}{2}}\Upsilon_{\varepsilon}t^{\frac{1}{2}}g_2(t)\,,F_2=Pf\left(\frac{d}{dt}\right)^{\frac{n-1}{2}}(1-\Upsilon_{-\varepsilon})|t|^{\frac{1}{2}}g_2(t)\,,\;F_3=\left(\frac{d}{dt}\right)^{\frac{n-1}{2}}g_1(t)\,.$$

Thus, the pair associated with every invariant distribution T of the equation $(\Box+k)T=0$ must be a linear combination of the four pairs $(F_1,0),(0,F_1),(F_2,F_2),(F_3,F_3)$.

The method developed in [6: ∮8] shows that the invariant distributions defined on $\mathbb{R}^n$ are as follows

$$T_1=Pf\;\;f_+^*\left(\frac{d}{dt}\right)^{\frac{n-1}{2}}\Upsilon_{\varepsilon}t^{\frac{1}{2}}g_2(t)\qquad,T_2=Pf\;\;f_\cdot^*\left(\frac{d}{dt}\right)^{\frac{n-1}{2}}(1-\Upsilon_{-\varepsilon})|t|^{\frac{1}{2}}g_2(t)$$

(10)

They are associated to $(F_1,0)$ and (F_2,F_2) respectively.

The transform $\overline{T}_1$ of T_1 under the improper Lorentz rotation is associated to $(0,F_1)$ and the distribution $T_3=\left(\frac{d}{dt}\right)^{\frac{n-1}{2}}g_1(t)$ continuous on $\mathbb{R}^n$ is associated to (F_3,F_3).

The distribution T is necessarily a linear combination of the distributions $T_1,\overline{T}_1,T_2$ and T_3 added to a distribution with support O.

Now let us see the ones which are linear combinations that satisfy
$(\Box+k)T=0$.
We have:

$$(\Box+k)T_1=Pf\;(\Box+k)\;f_+^*\left(\frac{d}{dt}\right)^{\frac{n-1}{2}}\Upsilon_{\varepsilon}t^{\frac{1}{2}}g_2(t)$$

$$=Pf\;\;f_+^*(L_n+k)\left(\frac{d}{dt}\right)^{\frac{1}{2}}\Upsilon_{\varepsilon}t^{\frac{1}{2}}g_2(t)$$

But from (9) and (10) we can write

$$(L_n+k)\left(\frac{d}{dt}\right)^{\frac{n-1}{2}}\Upsilon_{\varepsilon}t^{\frac{1}{2}}g_2(t)=\left(\frac{d}{dt}\right)^{\frac{n-1}{2}}(L_n+k)\Upsilon_{\varepsilon}t^{\frac{1}{2}}g_2(t)$$

$$=4\varepsilon^{\frac{3}{2}}g_2'(\varepsilon)\delta_{\varepsilon}^{\left(\frac{n-1}{2}\right)}+4\varepsilon^{\frac{3}{2}}g_2(\varepsilon)\delta_{\varepsilon}^{\left(\frac{n+1}{2}\right)}$$

This implies

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

$(\square + k)T_1 = 4Pf\varepsilon^{\frac{3}{2}} g_2'(\varepsilon) H_\varepsilon^{\frac{n-1}{2}} + 4Pf\varepsilon^{\frac{3}{2}} g_2(\varepsilon) H_\varepsilon^{\frac{n+1}{2}}$.
After many calculations, it is shown [6] that

$(\square + k)T_1 = (\square + k)\overline{T}_1 = \pi^{\frac{n-1}{2}} \delta_o$ and

$(\square + k)T_2 = (\square + k)T_3 = 0$

And taking $k = -\lambda$ this proves this theorem:

Theorem 7.2
If n is even $(n \geq 3)$ and $\lambda \neq 0$, the invariant general solution of the equation $(\square + \kappa)T = 0$, i.e. $\square T = \lambda T$ is $T = a(T_1 - \overline{T}_1) + bT_2 + cT_3$ where T_1 and T_2 are defined by (12) which $g_2(t) = \frac{\sin\sqrt{-\lambda t}}{\sqrt{-\lambda t}}$ and where $\overline{T}_1$ is the transform of T_1 under the improper Lorentz rotation and where T_3 is the distribution equal to the invariant function $\left(\frac{d}{dt}\right)^{\frac{n-1}{2}} \cos\sqrt{-\lambda t}$

The case n odd $(n \geq 4)$ similarly developed gives

$$T_1 = Pf\ f_+^* \left(\frac{d}{dt}\right)^{\frac{n-1}{2}} \Upsilon_\varepsilon h_1(t)$$

$$T_2 = Pf\ f^* \left(\frac{d}{dt}\right)^{\frac{n-2}{2}} (\Upsilon_\varepsilon + 1 - \Upsilon_{-\varepsilon}) h_2(t) \qquad (11)$$

And the following theorem follows:

Theorem 7.3
If n is odd $(n \geq 4)$ and $\lambda \neq 0$, then the general solution of equation $(\square - \lambda)T = 0$ is (with three arbitrary constants a, b and c):

$T = a(T_1 - \overline{T}_1) + bT_2 + cT_3$

where T_1 and T_2 are defined by (13) in which $h_1(t) = J_o(\sqrt{-\lambda t})$, J being the classical Bessel function and $h_2(t)$ is another linearly distinct solution of the equation $(L_2 - \lambda)V = 0$ and where $\overline{T}_1$ is the transform of T_1 under an improper Lorentz rotation and where

$$T_3 = \left(\frac{d}{dt}\right)^{\frac{n-2}{2}} J_o(\sqrt{-\lambda t}).$$

3. RESULTS

In this paper, solid foundations were established through the arguments on the hyperbolic space, the cone, representations and intertwining operators. Further, it is exposed that

Journal of Electronics, Computer Networking and Applied Mathematics
ISSN: 2799-1156
Vol: 02, No. 05, Aug-Sept 2022
http://journal.hmjournals.com/index.php/JECNAM
DOI: https://doi.org/10.55529/jecnam.25.34.57

spherical distributions can be derived using the Fourier transform and the hypergeometric function. In addition, we employed an effort to show that spherical distributions can be obtained by solving the equation $LS = \lambda S$ with the guide of Methėe's sequel [6].

4. CONCLUSION

Here, we established a relationship that translates into bringing out spherical distributions in a different facet. We deem fit to propose that spherical distributions can be expressed more simply when the underlying manifold is a compact Lie group.

5. REFERENCES

1. Erdėlyi A. et al, "Higher Transcendental Functions".Vol.1, McGraw-Hill, New York, 1953.
2. Faraut J., Distributions sphėriques sur les espaces hyperboliques, J. Math. Pure Appl. 5 (1979),369-444.
3. Helgason S.,"Differential Geometry and Symmetric Spaces", Academic Press, 1962.
4. Kosters M.T. and G.van Dijk, Spherical distributions on the pseudo-Riemannian spaces $SL(n,R)/GL(n-1,R)$. Journal of Functional Analysis 68(1986), 168-213.
5. Maire H.M., Sur les distributions image reciproque par une function analytique. Commentarii Math. Helvetici, 51(1976), 395-410.
6. Methėe P.D., Sur les distributions invariantes dans le groupe des rotations de Lorentz. Commentari Math. Helvetici,28 (1954), 225-269.
7. Patrick Delorme and Pascale Harninck, Relative Trace Formula for Compact Quotient and Pseudo-Coefficients for Relative Discrete Series. International Mathematics Research Notices. Volume 2022, Issue 7, 5506-5540, 2020.
8. Pushpendra Singh, Amit Singhal, Binish Fatimah, Anubha Gupta and Shiv Dutt Josh, On the Fourier Representations and Schwartz Distributions. Intitute of Electrical and Electronics Engineers(IEEE),2022.
9. G.de Rham, Solution ėlėmentaire d'opėrateurs differentiels du second ordre. Ann. Inst. Fourier, 8(1958), 337-366.
10. Schwartz L., Thėorie des distributions à valeurs vectorielles. Ann. Inst, Fourier. 7(1957), 1-141.
11. Tengstrand A., Distributions invariant under an orthogonal group of arbitrary signature. Math.Scand. 8(1960), 201-218.
12. Wallach N., "Harmonic Analysis on Homogeneous Spaces ". Dekker, New York, 1973.
13. Wolf J.A., Reprėsentations of certain semi-direct product groups; J. Function. Anal. 19(1975), 339-372.