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On Variance Components Estimators in Repeated Measurements Model

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Abstract: In this article, we focus our study on estimation the variance components in the two methods: the maximum likelihood function and the maximum penalized likelihood function of the repeated measurements model, which contain three random effects as well as random error, comparing between these estimators based on the mean square error , we also studied the bias and variance for each estimator and supported the theoretical side with an applied example to illustrate the results.

Keywords: Repeated Measurements Model (RMM), Random Effects, Maximum Likelihood (ML), Maximum Penalized Likelihood (MPL), Fixed Effect.

1. INTRODUCTION

Repeated measurements are data that show the response variable for each experimental unit many times and perhaps under various experimental settings [8]. Measurements that are repeated data are derived from experiments in which observations are made regularly over time. In the case of a repeated measurement experiment involves observing experimental units at various times in time. The experimental units are monitored at different times in time during the experiment. The analysis of repeated measurements is often used in many domains, including health and life sciences, epidemiological, agricultural, biomedical, industrial, psychological, and educational studies [8],[15],[22].

The maximum likelihood estimation is a method of estimating the coefficients of a statistical model and finding it for a set of data, by estimating the median of that model. When applied to a set of data and given a statistical model, the most likely probability estimate provides estimates for the model coefficients [19].

In theory of item responses, penalized likelihood model estimation has been utilized to produce better balanced estimates and to prevent boundary estimations in log-linear models,

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logistic regression, and latent variable analysis. This method has also been used to construct non-degenerate covariance matrices in finite mixtures of normal densities and multivariate regression. The penalized likelihood technique to ignoring boundary estimates for variance parameters in multilevel models is comparable to, but more broad than [5],[6],[7],[16],[17],[20].

The repeated measurements model has been studied extensively for instance: Mohaisen, A. J. and Abdulhussein discussed the fuzzy sets and penalized spline in Bayesian semiparametric regression,[18]. Yin and et al, they introduce a Bayesian procedure for the mixed-effects analysis of efficiency studies using mixed binomial regression models subjects in either one- or two-factor repeated-measures designs,[23]. AL-Mouel, Mohaisen and Khawla they are used Bayesian procedure based on Bayes quadratic unbiased estimator to the linear one - way repeated measurements model, [3]. AL-Mouel and Al-Isawi computed the quadratic unbiased estimator, which has minimum variance (best quadratic unbiased estimate),[4]. Innocent Ngaruye, Dietrich von Rosen and Martin Singull discussed the mean-squared errors of small area estimators under a multivariate linear model for repeated measures data,[9]. Jassim and AL-Mouel they propose the lasso method for choice of penalty level and investigate the error of the lasso estimator in repeated measurements model,[10]. AL-Mouel and Kori in (2021) studied estimating the parameters and properties of the repeated measurement model in two cases: conditional and unconditional, [2],[11],[12],[13],[14].

In this article, we will focus our study on estimating the variance components in the two methods: the maximum likelihood function and the maximum penalized likelihood function of the repeated measurements model, which contain three random effects as well as random error, comparing them based on mean square error , we will work on applying an applied example to support and prove the theoretical aspect of this work to obtain identical results.

Formulation the Model

$$\Psi_{npq} = \mu + \xi_p + \eta_q + (\xi\eta)_{pq} + \theta_{n(p)} + \lambda_{n(q)} + \varphi_{n(pq)} + e_{nqp} \quad (1)$$

where Ψ_{npq} is the response variable, ξ_p and η_q are the fixed effects for all $p = 1, \dots, v$ and $q = 1, \dots, s$ repectively, $\theta_{n(p)}$, $\lambda_{n(q)}$ *and* $\varphi_{n(pq)}$ are the random effects for all $n = 1, \dots, u$, $p = 1, \dots, v$ and $q = 1, \dots, s$ and e_{nqp} is a random error for all $n = 1, \dots, u$, $p = 1, \dots, v$ and $q = 1, \dots, s$.

$\sum_{p=1}^{u} \xi_p = 0$; $\sum_{q=1}^{v} \eta_q = 0$; $\sum_{p=1}^{v} (\xi\eta)_{pq} = 0$ for each $q = 1, \dots, s$;
$\sum_{q=1}^{s} (\xi\eta)_{pq} = 0$ for each $q = 1, \dots, s$.
And we assume that the random parameters and random error are independent with
$\theta_{n(p)}\ i.i.d \sim N(0, \sigma_\theta^2)$, $\lambda_{n(q)}\ i.i.d \sim N(0, \sigma_\lambda^2)$, $\varphi_{n(pq)} i.i.d \sim N(0, \sigma_\varphi^2)$ and
$\varepsilon_{xyz}\ i.i.d \sim N(0, \sigma_e^2)$. (2)
The analysis of the variance table of the repeated measurements model included the sum of squares terms, which can be represented as follow.

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Table (1): the ANOVA table of repeated measurements model

Source of variance	**D.F.**	**SS**	**MS**	**E(MS)**
Tretment ξ	$v-1$	SS_ξ	$\frac{MS_\xi}{v-1}$	$\frac{us}{s-1}\sum_{p=1}^{v}\xi_p^2+s\sigma_\theta^2+\sigma_\varphi^2+\sigma_e^2$
treatment η	$s-1$	SS_η	$\frac{MS_\eta}{s-1}$	$\frac{uv}{u-1}\sum_{q=1}^{s}\eta_q^2+p\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2$
Intrection $\xi\times\eta$	$(v-1)(s-1)$	$SS_{\xi\times\eta}$	$\frac{MS_{\xi\times\eta}}{(v-1)(s-1)}$	$\frac{u}{(s-1)(v-1)}\sum_{p=1}^{v}\sum_{q=1}^{s}(\xi\eta)_{pq}^2+\sigma_e^2$
Random θ	$v(u-1)$	SS_θ	$\frac{MS_\theta}{v(u-1)}$	$s\sigma_\theta^2+\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2$
Random λ	$s(u-1)$	SS_λ	$\frac{MS_\lambda}{s(u-1)}$	$\sigma_\theta^2+v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2$
Random φ	$vs(u-1)$	SS_φ	$\frac{MS_\varphi}{vs(u-1)}$	$\sigma_\theta^2+\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2$
Error e	$(v+s)(1-u)$	SS_e	MS_E	σ_e^2
$\mathbf{SS_{total}}$	**$uvs-1$**			

Where,
$SS_\xi = us\sum_{p=1}^{v}(\bar{\Psi}_{.p.}-\Psi_{...})^2$, $SS_\eta = uv\sum_{v=1}^{s}(\bar{\Psi}_{..v}-\bar{\Psi}_{...})^2$,
$SS_{\xi\times\eta} = u\sum_{n=1}^{u}\sum_{p=1}^{v}\sum_{v=1}^{s}(\bar{\Psi}_{.pq}-\bar{\Psi}_{.p.}-\bar{\Psi}_{..q}+\bar{\Psi}_{...})^2$,
$SS_\theta = s\sum_{n=1}^{u}\sum_{p=1}^{v}(\bar{\Psi}_{np.}-\bar{\Psi}_{.p.})^2$, $SS_\lambda = v\sum_{n=1}^{u}\sum_{q=1}^{s}(\bar{\Psi}_{n.q}-\bar{\Psi}_{..q})^2$,
$SS_\varphi = \sum_{n=1}^{u}\sum_{p=1}^{v}\sum_{q=1}^{s}(\bar{\Psi}_{npq}-\bar{\Psi}_{.pq})^2$ and
$SS_e = \sum_{n=1}^{u}\sum_{p=1}^{v}\sum_{q=1}^{s}(\bar{\Psi}_{.p.}+\bar{\Psi}_{..k}-\bar{\Psi}_{np.}-\bar{\Psi}_{n.q})^2$

Where,
$\bar{\Psi}_{...} = \frac{1}{uvs}\sum_{n=1}^{u}\sum_{p=1}^{v}\sum_{q=1}^{s}\Psi_{npq}$, $\bar{\Psi}_{..q} = \frac{1}{uv}\sum_{n=1}^{u}\sum_{p=1}^{v}\Psi_{npq}$,
$\bar{\Psi}_{.p.} = \frac{1}{us}\sum_{n=1}^{u}\sum_{q=1}^{s}\Psi_{npq}$, $\bar{\Psi}_{.pq} = \frac{1}{u}\sum_{n=1}^{u}\Psi_{npq}$,
$\bar{\Psi}_{n.q} = \frac{1}{v}\sum_{p=1}^{v}\Psi_{npq}$ and $\bar{\Psi}_{np.} = \frac{1}{s}\sum_{q=1}^{s}\Psi_{npq}$.

The distribution of sum square as follows:
$SS_e \sim \sigma_e^2\chi^2((v+s)(1-u))$, $SS_\xi \sim (s\sigma_\theta^2+\sigma_\varphi^2+\sigma_e^2)\chi^2(v-1)$,
$SS_\eta \sim (v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2)\chi^2(vs-1)$, $SS_{\xi\times\eta} \sim (\sigma_\varphi^2+\sigma_e^2)\chi^2(v(v-1)(s-1))$,
$SS_\theta \sim (s\sigma_\theta^2+\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2)\chi^2(v(u-1))$, $SS_\lambda \sim (\sigma_\theta^2+v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2)\chi^2(s(u-1))$
and $SS_\varphi \sim (\sigma_\theta^2+\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2)\chi^2(vs(u-1))$.

Maximum Likelihood Estimator of Variance Components

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the likelihood function for the model (1) as follow:

$$L(\Psi) = \frac{exp\left\{-\frac{1}{2}\Omega\right\}}{\Lambda} \tag{3}$$

where

$$\Omega = \frac{SS_e}{\sigma_e^2} + \frac{SS_\xi}{s\sigma_\theta^2+\sigma_\varphi^2+\sigma_e^2} + \frac{SS_\eta}{v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2} + \frac{SS_{\xi\times\eta}}{\sigma_\varphi^2+\sigma_e^2} + \frac{SS_\theta}{s\sigma_\theta^2+\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2} + \frac{SS_\lambda}{\sigma_\theta^2+v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2} + \frac{SS_\varphi}{\sigma_\theta^2+\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2} + \frac{uvs(\bar{\Psi}_{...}-\mu)^2}{s\sigma_\theta^2+v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2}$$

and

$$\Lambda = (2\pi)^{\frac{1}{2}npq}(\sigma_e^2)^{\frac{1}{2}(1-u)(v+s)}(s\sigma_\theta^2+\sigma_\varphi^2+\sigma_e^2)^{\frac{1}{2}(v-1)}(v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2)^{\frac{1}{2}(s-1)}(\sigma_\varphi^2+\sigma_e^2)^{\frac{1}{2}(v-1)(s-1)}\left(s\sigma_\theta^2+\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2\right)^{\frac{1}{2}v(u-1)}\left(\sigma_\theta^2+v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2\right)^{\frac{1}{2}s(u-1)}(\sigma_\theta^2+\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2)^{\frac{1}{2}vs(u-1)}\left(s\sigma_\theta^2+v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2\right)^{\frac{1}{2}}$$

The log-likelihood function of (3) can be written as:

$$Ln\ L(\Psi) = -\frac{1}{2}\Big[npq\ ln(2\pi) + (1-u)(v+s)\ln(\sigma_e^2) + (v-1)\ln\left(s\sigma_\theta^2+\sigma_\varphi^2+\sigma_e^2\right) + (s-1)ln\left(v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2\right) + (v-1)(s-1)ln\left(\sigma_\varphi^2+\sigma_e^2\right) + v(u-1)ln\left(s\sigma_\theta^2+\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2\right) + s(u-1)\ ln\left(\sigma_\theta^2+v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2\right) + vs(u-1)ln\left(\sigma_\theta^2+\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2\right) + \ln\left(s\sigma_\theta^2+v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2\right) + \frac{SS_e}{\sigma_e^2} + \frac{SS_\xi}{s\sigma_\theta^2+\sigma_\varphi^2+\sigma_e^2} + \frac{SS_\eta}{v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2} + \frac{SS_{\xi\times\eta}}{\sigma_\varphi^2+\sigma_e^2} + \frac{SS_\theta}{s\sigma_\theta^2+\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2} + \frac{SS_\lambda}{\sigma_\theta^2+v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2} + \frac{SS_\varphi}{\sigma_\theta^2+\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2} + \frac{uvs(\bar{\Psi}_{...}-\mu)^2}{s\sigma_\theta^2+v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2}\Big] \tag{4}$$

Now, we will maximize the equation (4) with respect to σ_e^2 and equal to zero, we have that

$$\frac{\partial Ln\ L(\Psi)}{\partial\sigma_e^2} = -\frac{1}{2}\left[\frac{(1-u)(v+s)}{\sigma_e^2} - \frac{SS_e}{\sigma_e^4}\right] = 0$$

$$\frac{(1-u)(v+s)}{\sigma_e^2} - \frac{SS_e}{\sigma_e^4} = 0$$

$$(1-u)(v+s)\sigma_e^2 - SS_e = 0$$

$$\therefore\ \hat{\sigma}_e^2 = \frac{SS_e}{(1-u)(v+s)} = MS_E \tag{5}$$

Now, we will maximize the equation (4) with respect to $\left(\sigma_\varphi^2+\sigma_e^2\right)$ and equal to zero, we have that

$$\frac{\partial Ln\ L(\Psi)}{\partial(\sigma_\varphi^2+\sigma_e^2)} = -\frac{1}{2}\left[\frac{(v-1)(s-1)}{\sigma_\varphi^2+\sigma_e^2} - \frac{SS_{\xi\times\eta}}{\left(\sigma_\varphi^2+\sigma_e^2\right)^2}\right] = 0$$

$$\hat{\sigma}_\varphi^2 = \frac{SS_{\xi\times\eta}}{(v-1)(s-1)} - \hat{\sigma}_e^2$$

$$\therefore\ \hat{\sigma}_\varphi^2 = \frac{SS_{\xi\times\eta}}{(v-1)(s-1)} - \hat{\sigma}_e^2 = MS_{\xi\times\eta} - MS_E \tag{6}$$

Now, we will maximize the equation (4) with respect to $\left(s\sigma_\theta^2+\sigma_\varphi^2+\sigma_e^2\right)$ and equal to zero, we have that

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$$\frac{\partial Ln\, L(\Psi)}{\partial(s\sigma_\theta^2+\sigma_\varphi^2+\sigma_e^2)} = -\frac{1}{2}\left[\frac{(v-1)}{s\sigma_\theta^2+\sigma_\varphi^2+\sigma_e^2} - \frac{sSS_\theta}{(s\sigma_\theta^2+\sigma_\varphi^2+\sigma_e^2)^2}\right] = 0$$

$$\hat{\sigma}_\theta^2 = \frac{SS_\theta - \hat{\sigma}_\varphi^2 - \hat{\sigma}_e^2}{s}$$

$$\therefore\ \hat{\sigma}_\theta^2 = \frac{MS_\theta - MS_{\xi\times\eta} - MS_E}{s} \quad (7)$$

Now, we will maximize the equation (4) with respect to $(v\sigma_\lambda^2 + \sigma_\varphi^2 + \sigma_e^2)$ and equal to zero, we have that

$$\frac{\partial Ln\, L(\Psi)}{\partial(v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2)} = -\frac{1}{2}\left[\frac{(s-1)}{v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2} - \frac{SS_\eta}{(v\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2)^2}\right] = 0$$

$$\hat{\sigma}_\lambda^2 = \frac{(SS_\eta/s-1) - \hat{\sigma}_\varphi^2 - \hat{\sigma}_e^2}{v}$$

$$\therefore\ \hat{\sigma}_\lambda^2 = \frac{MS_\eta - MS_{\xi\times\eta} - MS_E}{v} \quad (8)$$

Theorem 1: The maximum likelihood estimators of variance components are unbiased.
Proof:

$$E(\hat{\sigma}_e^2) - \sigma_e^2 = E\left(\frac{SS_e}{(1-u)(v+s)}\right) - \hat{\sigma}_e^2 = \frac{1}{(1-u)(v+s)}E(SS_e) - \hat{\sigma}_e^2$$
$$= \frac{1}{(1-u)(v+s)}\big((1-u)(v+s)\big)\hat{\sigma}_e^2 - \hat{\sigma}_e^2 = 0$$

$$E(\hat{\sigma}_\varphi^2) - \sigma_\varphi^2 = E\left[\frac{SS_{\xi\times\eta}}{(v-1)(s-1)} - \hat{\sigma}_e^2\right] - \sigma_\varphi^2 = \frac{1}{(v-1)(s-1)}E\left[SS_{\xi\times\eta} - \hat{\sigma}_e^2\right] - \sigma_\varphi^2$$
$$= \frac{1}{(v-1)(s-1)}E\left[SS_{\xi\times\eta} - \hat{\sigma}_e^2\right] - \sigma_\varphi^2 = \frac{1}{(v-1)(s-1)}\left[\big((v-1)(s-1)\big)(\hat{\sigma}_\varphi^2 + \hat{\sigma}_e^2 - \hat{\sigma}_e^2)\right] - \sigma_\varphi^2 = \sigma_\varphi^2 + \hat{\sigma}_e^2 - \hat{\sigma}_e^2 - \sigma_\varphi^2 = 0$$

$$E(\hat{\sigma}_\theta^2) - \sigma_\theta^2 = E\left[\frac{SS_\theta - \hat{\sigma}_\varphi^2 - \hat{\sigma}_e^2}{s}\right] - \sigma_\theta^2 = \frac{1}{s}\left[s(\sigma_\theta^2)\right] - \sigma_\theta^2 = 0$$

$$E(\hat{\sigma}_\lambda^2) - \sigma_\lambda^2 = E\left[\frac{SS_\eta}{s-1} - \hat{\sigma}_\varphi^2 - \hat{\sigma}_e^2\right] - \sigma_\lambda^2 = \frac{1}{s-1}E\left[SS_\eta\right] - E\left[\hat{\sigma}_\varphi^2 + \hat{\sigma}_e^2\right] - \sigma_\theta^2$$
$$= \frac{1}{s-1}(s-1)(\sigma_\lambda^2) - \sigma_\lambda^2 = 0$$

Theorem 2: The variance of maximum likelihood estimators are biased.

$$var(\hat{\sigma}_e^2) = var\left(\frac{SS_e}{(1-u)(v+s)}\right) = \frac{var(SS_e)}{\big((1-u)(v+s)\big)^2} = \frac{(v+s)(1-u)\sigma_e^4}{\big((1-u)(v+s)\big)^2} = \frac{\sigma_e^4}{(1-u)(v+s)}$$

$$var(\hat{\sigma}_\varphi^2) = var\left(\frac{SS_{\xi\times\eta}}{(v-1)(s-1)} - \hat{\sigma}_e^2\right) = \frac{var(SS_{\xi\times\eta})}{\big((v-1)(s-1)\big)^2} + var(\hat{\sigma}_e^2)$$
$$= \frac{(v-1)(s-1)\sigma_e^4}{\big((v-1)(s-1)\big)^2} + \frac{\sigma_e^4}{(1-u)(v+s)} = \left[\frac{1}{(v-1)(s-1)} + \frac{1}{(1-u)(v+s)}\right]\sigma_e^4$$

$$var(\hat{\sigma}_\theta^2) = var\left(\frac{SS_\theta - \hat{\sigma}_\varphi^2 - \hat{\sigma}_e^2}{s}\right) = \frac{var(SS_\theta - \hat{\sigma}_\varphi^2 - \hat{\sigma}_e^2)}{s^2} = \frac{1}{s^2}\left[\frac{(s\sigma_\theta^2 + \sigma_\lambda^2 + \sigma_\varphi^2 + \sigma_e^2)^2}{v(u-1)} + \left(\frac{1}{(v-1)(s-1)} + \frac{1}{(1-u)(v+s)}\right)\sigma_e^4\right]$$

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$$var(\hat{\sigma}_{\lambda}^{2}) = var\left(\frac{(SS_{\eta}/s-1)-\hat{\sigma}_{\varphi}^{2}-\hat{\sigma}_{e}^{2}}{v}\right) = \frac{\frac{var(SS_{\eta})}{(s-1)^{2}}+var(\hat{\sigma}_{\varphi}^{2}-\hat{\sigma}_{e}^{2})}{v^{2}} = \frac{v^{2}(p\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{\varepsilon}^{2})^{2}}{(s-1)} + \left(\frac{v^{2}}{(v-1)(s-1)}+\frac{v^{2}}{(1-u)(v+s)}\right)\sigma_{e}^{4}$$

Maximum Penalized Likelihood Estimators

The maximum penalized likelihood function can be written as follows:

$$L = \Psi + p(\sigma_{\theta}^{2},\sigma_{\lambda}^{2},\sigma_{\varphi}^{2},\sigma_{e}^{2}) \tag{9}$$

"the penalized logarithm-likelihood function can be written as follows:"

$$\ln(L) = L(\Psi) + \ln p(\sigma_{\theta}^{2},\sigma_{\lambda}^{2},\sigma_{\varphi}^{2},\sigma_{e}^{2}) \tag{10}$$

where

$L(\Psi)$ the logarithm-likelihood and $\ln p(\sigma_{\theta}^{2},\sigma_{\lambda}^{2},\sigma_{\varphi}^{2},\sigma_{e}^{2})$ the penalty term.

The logarithm-likelihood for the Bayesian prior density is used to calculate the added penalty. Tiao and Tan (1965) supposed about the prior distribution for $\mu,\sigma_{\theta}^{2},\sigma_{\lambda}^{2},\sigma_{\varphi}^{2},\sigma_{e}^{2}$ as follows: [21]

$$\mathrm{p}(\sigma_{\theta}^{2},\sigma_{\lambda}^{2},\sigma_{\varphi}^{2},\sigma_{e}^{2}) \propto \frac{1}{\sigma_{e}^{2}(s\sigma_{\theta}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2})(v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2})(\sigma_{\varphi}^{2}+\sigma_{e}^{2})} \tag{11}$$

$$\ln L(\Psi) = -\frac{1}{2}\Big[npq\,ln(2\pi) + (1-u)(v+s)\ln(\sigma_{e}^{2}) + (v-1)\ln(s\sigma_{\theta}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}) + (s-1)ln(v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}) + (v-1)(s-1)ln(\sigma_{\varphi}^{2}+\sigma_{e}^{2}) + v(u-1)ln(s\sigma_{\theta}^{2}+\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}) + s(u-1)\ln(\sigma_{\theta}^{2}+v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}) + vs(u-1)ln(\sigma_{\theta}^{2}+\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}) + \ln(s\sigma_{\theta}^{2}+v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}) + \frac{SS_{e}}{\sigma_{e}^{2}} + \frac{SS_{\xi}}{s\sigma_{\theta}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}} + \frac{SS_{\eta}}{v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}} + \frac{SS_{\xi\times\eta}}{\sigma_{\varphi}^{2}+\sigma_{e}^{2}} + \frac{SS_{\theta}}{s\sigma_{\theta}^{2}+\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}} + \frac{SS_{\lambda}}{\sigma_{\theta}^{2}+v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}} + \frac{SS_{\varphi}}{\sigma_{\theta}^{2}+\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}} + \frac{uvs(\bar{\Psi}_{\dots}-\mu)^{2}}{s\sigma_{\theta}^{2}+v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}}\Big] - \ln(\sigma_{e}^{2}) - \ln(s\sigma_{\theta}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}) - \ln(v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}) - \ln(\sigma_{\varphi}^{2}+\sigma_{e}^{2}) \tag{12}$$

we will maximize the equation (12) with respect to (σ_{e}^{2}), $(\sigma_{\varphi}^{2}+\sigma_{e}^{2})$, $(s\sigma_{\theta}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2})$ *and* $(v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2})$, equal to zero, we have that

$$\frac{\partial Ln\,L(\Psi)}{\partial\sigma_{e}^{2}} = -\frac{1}{2}\left[\frac{(1-u)(v+s)}{\sigma_{e}^{2}} - \frac{SS_{e}}{\sigma_{e}^{4}} + \frac{1}{\sigma_{e}^{2}}\right] = 0 \rightarrow \hat{\sigma}_{e}^{2} = \frac{SS_{e}}{(1-u)(v+s)+1} \tag{13}$$

$$\frac{\partial Ln\,L(\Psi)}{\partial(\sigma_{\varphi}^{2}+\sigma_{e}^{2})} = -\frac{1}{2}\left[\frac{(v-1)(s-1)}{\sigma_{\varphi}^{2}+\sigma_{e}^{2}} - \frac{SS_{\xi\times\eta}}{(\sigma_{\varphi}^{2}+\sigma_{e}^{2})^{2}} + \frac{1}{\sigma_{\varphi}^{2}+\sigma_{e}^{2}}\right] = 0$$

$$\rightarrow \hat{\sigma}_{\varphi}^{2} = \frac{SS_{\xi\times\eta}-\hat{\sigma}_{e}^{2}[(v-1)(s-1)+1]}{(v-1)(s-1)+1} \tag{14}$$

$$\frac{\partial Ln\,L(\Psi)}{\partial(s\sigma_{\theta}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2})} = -\frac{1}{2}\left[\frac{(v-1)}{s\sigma_{\theta}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}} - \frac{sSS_{\theta}}{(s\sigma_{\theta}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2})^{2}} + \frac{1}{s\sigma_{\theta}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}}\right] = 0$$

$$\rightarrow \hat{\sigma}_{\theta}^{2} = \frac{sSS_{\theta}-(s(u-1)+1)(\hat{\sigma}_{\varphi}^{2}+\hat{\sigma}_{e}^{2})}{s[(v-1)+1]} \tag{15}$$

$$\frac{\partial Ln\,L(\Psi)}{\partial(v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2})} = -\frac{1}{2}\left[\frac{v(s-1)}{v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}} - \frac{SS_{\eta}}{(v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2})^{2}} + \frac{1}{v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}}\right] = 0$$

$$\frac{v(s-1)}{v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}} - \frac{SS_{\eta}}{(v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2})^{2}} + \frac{1}{v\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}} = 0$$

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$$\rightarrow \hat{\sigma}_{\lambda}^{2} = \frac{SS_{\eta}-[(s-1)+1]-(\hat{\sigma}_{\varphi}^{2}+\hat{\sigma}_{e}^{2})}{v[(s-1)+1]} \quad (16)$$

Theorem 3: the maximum penalized likelihood estimators of variance components for model (1) are biased.

Proof:

$$E(\hat{\sigma}_{e}^{2}) - \sigma_{e}^{2} = E\left(\frac{SS_{e}}{(1-u)(v+s)+1}\right) - \hat{\sigma}_{e}^{2} = \frac{1}{(1-u)(v+s)}E(SS_{e}) - \hat{\sigma}_{e}^{2}$$

$$= \frac{1}{(1-u)(v+s)+1}\big((1-u)(v+s)\big)\hat{\sigma}_{e}^{2} - \hat{\sigma}_{e}^{2} \quad E(\hat{\sigma}_{\varphi}^{2}) - \sigma_{\varphi}^{2} =$$

$$E\left[\frac{SS_{\xi\times\eta}-\hat{\sigma}_{e}^{2}[(v-1)(s-1)+1]}{(v-1)(s-1)+1}\right] - \sigma_{\varphi}^{2}$$

$$= \frac{1}{(v-1)(s-1)+1}\left(E[SS_{\xi\times\eta}] - \big((v-1)(s-1)\big)E[\hat{\sigma}_{e}^{2}]\right) - \sigma_{\varphi}^{2}$$

$$= \frac{1}{(v-1)(s-1)+1}E\left[SS_{\xi\times\eta} - \hat{\sigma}_{e}^{2}\right] - \sigma_{\varphi}^{2}$$

$$= \frac{1}{(v-1)(s-1)+1}\left[\big((v-1)(s-1)+1\big)(\hat{\sigma}_{\varphi}^{2}+\hat{\sigma}_{e}^{2}) - \big((v-1)(s-1)\big)\hat{\sigma}_{e}^{2}\right] - \sigma_{\varphi}^{2}$$

$$E(\hat{\sigma}_{\theta}^{2}) - \sigma_{\theta}^{2} = E\left[\frac{sSS_{\theta}-(s(u-1)+1)(\hat{\sigma}_{\varphi}^{2}+\hat{\sigma}_{e}^{2})}{s[s(u-1)+1]}\right] - \sigma_{\theta}^{2} = \frac{sE[SS_{\theta}]-(s(u-1)+1)(E[\hat{\sigma}_{\varphi}^{2}+\hat{\sigma}_{e}^{2}])}{s[s(u-1)+1]}$$

$$= \frac{s(\sigma_{\varphi}^{2}+\sigma_{e}^{2})-(s(u-1)+1)(E[\hat{\sigma}_{\varphi}^{2}+\hat{\sigma}_{e}^{2}])}{s[s(u-1)+1]} - \sigma_{\theta}^{2}$$

$$E(\hat{\sigma}_{\lambda}^{2}) - \sigma_{\lambda}^{2} = E\left[\frac{SS_{\eta}-[v(s-1)+1]-(\hat{\sigma}_{\varphi}^{2}+\hat{\sigma}_{e}^{2})}{v[v(s-1)+1]}\right] - \sigma_{\lambda}^{2} = \frac{1}{v[v(s-1)+1]}E\left([SS_{\eta}] - E[\hat{\sigma}_{\varphi}^{2}+\hat{\sigma}_{e}^{2}]\right) - \sigma_{\lambda}^{2}$$

$$= \frac{1}{v[v(s-1)+1]}E\left((s-1)(\sigma_{\lambda}^{2}) - E[\hat{\sigma}_{\varphi}^{2}+\hat{\sigma}_{e}^{2}]\right) - \sigma_{\lambda}^{2}$$

Theorem 4: the variance of maximum penalized likelihood estimators as follows.

$$var(\hat{\sigma}_{e}^{2}) = var\left(\frac{SS_{e}}{(1-u)(v+s)+1}\right) = \frac{var(SS_{e})}{\big((1-u)(v+s)+1\big)^{2}} = \frac{(v+s)(1-u)\sigma_{e}^{4}}{\big((1-u)(v+s)+1\big)^{2}} = \frac{(v+s)(1-u)\sigma_{e}^{4}}{\big((1-u)(v+s)+1\big)^{2}}$$

$$var(\hat{\sigma}_{\varphi}^{2}) = var\left(\frac{SS_{\xi\times\eta}-\hat{\sigma}_{e}^{2}[(v-1)(s-1)+2]}{(v-1)(s-1)+1}\right) = \frac{var(SS_{\xi\times\eta})+[(v-1)(s-1)+2]var(\hat{\sigma}_{e}^{2})}{\big((v-1)(s-1)+1\big)^{2}}$$

$$= \frac{(v-1)(s-1)\sigma_{e}^{4}+[(v-1)(s-1)+1]v\frac{(v+s)(1-u)\sigma_{e}^{4}}{\big((1-u)(v+s)+1\big)^{2}}}{\big((v-1)(s-1)+1\big)^{2}}$$

$$var(\hat{\sigma}_{\theta}^{2}) = var\left(\frac{sSS_{\theta}-(s(u-1)+1)(\hat{\sigma}_{\varphi}^{2}+\hat{\sigma}_{e}^{2})}{s[s(u-1)+1]}\right) = \frac{s\,var(SS_{\theta})+(s(u-1)+1))\,var)(^{2}_{\varphi}-\hat{\sigma}_{e}^{2})}{s[s(u-1)+1]^{2}} =$$

$$\frac{1}{s^{2}}\left[\frac{\big(s\sigma_{\theta}^{2}+\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}\big)^{2}+(s(u-1)+1)^{2}(x+y)}{s[s(u-1)+1]^{2}}\right]$$

where $x = var(\hat{\sigma}_{e}^{2})$ and $y = var(\hat{\sigma}_{\varphi}^{2})$

$$var(\hat{\sigma}_{\lambda}^{2}) = var\left(\frac{SS_{\eta}-[v(s-1)+1](\hat{\sigma}_{\varphi}^{2}+\hat{\sigma}_{e}^{2})}{v[v(s-1)+1]}\right) = \frac{var(SS_{\eta})+[v(s-1)+1]^{2}var(\hat{\sigma}_{\varphi}^{2}-\hat{\sigma}_{e}^{2})}{v[v(s-1)+1]} =$$

$$\frac{\big(p\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{e}^{2}\big)^{2}+[(s-1)+1]^{2}(x+y)}{v[(s-1)+1]}$$

where $x = var(\hat{\sigma}_{e}^{2})$ and $y = var(\hat{\sigma}_{\varphi}^{2})$.

Difference between Variance Components Estimators

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The next theorem has been used to demonstrate the link among variance component estimators and compare these estimators employing mean square error.

Theorem 5: The mean square error of variance components of model (1) are as follows:
$MSE\left(\hat{\sigma}^2_{e,ML}\right) \leq MSE\left(\hat{\sigma}^2_{e,MPL}\right)$,
$MSE\left(\hat{\sigma}^2_{\varphi,ML}\right) \leq MSE\left(\hat{\sigma}^2_{\varphi,MPL}\right)$,
$MSE\left(\hat{\sigma}^2_{\theta,ML}\right) \leq MSE\left(\hat{\sigma}^2_{\theta,MPL}\right)$ and
$MSE\left(\hat{\sigma}^2_{\lambda,ML}\right) \leq MSE\left(\hat{\sigma}^2_{\lambda,MPL}\right)$.

Poof:

$$MSE\left(\hat{\sigma}^2_{e,ML}\right) = \frac{\hat{\sigma}^4_e}{(1-u)(v+s)} \leq MSE\left(\hat{\sigma}^2_{e,MPL}\right) = \frac{(v+s)(1-u)\hat{\sigma}^4_e}{\left((1-u)(v+s)+1\right)^2} + \left(\frac{1}{(1-u)(v+s)+1}\left((1-u)(v+s)\right)\hat{\sigma}^2_e - \hat{\sigma}^2_e\right)^2,$$

$$MSE\left(\hat{\sigma}^2_{\varphi,ML}\right) = \left[\frac{1}{(v-1)(s-1)} + \frac{1}{(1-u)(v+s)}\right]\sigma^4_e \leq MSE\left(\hat{\sigma}^2_{\varphi,MPL}\right) = \frac{(v-1)(s-1)\sigma^4_e + [(v-1)(s-1)+1]v\frac{(v+s)(1-u)\sigma^4_e}{\left((1-u)(v+s)+1\right)^2}}{\left((v-1)(s-1)+1\right)^2} + \left(\frac{1}{(v-1)(s-1)+1}\left[\left((v-1)(s-1)+1\right)(\hat{\sigma}^2_\varphi + \hat{\sigma}^2_e) - \left((v-1)(s-1)\right)\hat{\sigma}^2_e\right] - \hat{\sigma}^2_\varphi\right)^2,$$

$$MSE\left(\hat{\sigma}^2_{\theta,ML}\right) = \frac{1}{s^2}\left[\frac{\left(s\sigma^2_\theta+\sigma^2_\lambda+\sigma^2_\varphi+\sigma^2_e\right)^2}{v(u-1)} + \left(\frac{1}{(v-1)(s-1)} + \frac{1}{(1-u)(v+s)}\right)\sigma^4_e\right] \leq MSE\left(\hat{\sigma}^2_{\theta,MPL}\right) = \left[\frac{\left(s\sigma^2_\theta+\sigma^2_\lambda+\sigma^2_\varphi+\sigma^2_e\right)^2 + (s(u-1)+1)^2(x+y)}{s[s(u-1)+1]^2}\right] + \left(\frac{s(\sigma^2_\varphi+\sigma^2_e) - (s(u-1)+1)\left(E[\hat{\sigma}^2_\varphi+\hat{\sigma}^2_e]\right)}{s[s(u-1)+1]} - \sigma^2_\theta\right)^2$$

$$MSE\left(\hat{\sigma}^2_{\lambda,ML}\right) = \frac{\left(v\sigma^2_\lambda+\sigma^2_\varphi+\sigma^2_e\right)^2}{(s-1)} + \left(\frac{1}{(v-1)(s-1)} + \frac{1}{(1-u)(v+s)}\right)\sigma^4_e \leq MSE\left(\hat{\sigma}^2_{\lambda,MPL}\right) = \frac{\left(v\sigma^2_\lambda+\sigma^2_\varphi+\sigma^2_e\right)^2 + [v(s-1)+1]^2(x+y)}{v[v(s-1)+1]} + \left(\frac{1}{v[v(s-1)+1]}E\left(\left[SS_\eta\right] - E\left[\hat{\sigma}^2_\varphi + \hat{\sigma}^2_e\right]\right) - \sigma^2_\lambda\right)^2.$$

The Practical Side

In this section we chose an experiment to demonstrate the practical side of the methods used in our study.
The data set was taken from an experiment conducted in a private orchard in Abu Al-Khasib in Basra governorate during the two agricultural seasons 2018 and 2019 in order to study the effect of the season, plant variety, levels of sculpture, two levels of superphosphate and levels of soil leaching process on the height of two lettuce cultivars grown in highly saline soils,[1].
We take the data set from an experiment conducted in a private orchard in Abi Al-Khasib, Basrah Province, during the two agricultural seasons 2018 and 2019. The purpose of our application is to examine the effect of season, plant variety, different levels of sculpture as well as two different levels of superphosphate and levels of soil leaching process on the height of lettuce plants grown in highly saline environment. The experiment included 360 variable treatments, namely the season, two types of plants (Fajer and local), sulphur at levels 0, 500, 1000, 1500 and 2000 mm, superphosphate at levels of 200 and 400 mm, and levels of

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soil washing process. The experiment was designed according to the analysis of variance of the repeated measurements model. As shown in the ANOVA table (2).

Table (2): the ANOVA table of repeated measurements model

Source of variance	D.F.	SS	MS	F-Test
Season	1	116.463	116.463	183.482
Cultivar	1	76.176	76.176	120.012
Season × Cultivar	1	29.722	29.722	46.825
Sulpher	4	4057.859	1014.456	1598.242
Superphosphate	1	350.069	350.069	551.518
Soil Washing	2	236.524	118.189	186.201
Residual	349	221.524	0.634	
Total	359	5088.191		

The maximum likelihood estimators of variance components as follows:
$\hat{\sigma}_\theta^2 = 85.471, \hat{\sigma}_\lambda^2 = 45.184, \hat{\sigma}_\varphi^2 = 29.087$ and $\hat{\sigma}_e^2 = 0.634$,
$MSE\left(\hat{\sigma}_{e,ML}^2\right) = 0.00037$
$MSE\left(\hat{\sigma}_{\varphi,ML}^2\right) = 0.1328$
$MSE\left(\hat{\sigma}_{\theta,ML}^2\right) = 10471.0912$ and
$MSE\left(\hat{\sigma}_{\lambda,ML}^2\right) = 1782.$

The maximum penalized likelihood estimators of variance components as follows:
$\hat{\sigma}_\theta^2 = 2009.471, \hat{\sigma}_\lambda^2 = 59.315, \hat{\sigma}_\varphi^2 = 14.499$ and $\hat{\sigma}_e^2 = 0.632.$
$MSE\left(\hat{\sigma}_{e,MPL}^2\right) = 0.001$
$MSE\left(\hat{\sigma}_{\varphi,MPL}^2\right) = 49.998225$
$MSE\left(\hat{\sigma}_{\theta,MPL}^2\right) = 401452.322$ and
$MSE\left(\hat{\sigma}_{\lambda,MPL}^2\right) = 342167.876.$

We see that the value of maximum likelihood estimators and man square error of variance components less than the mean square error and maximum penalized likelihood estimators of variance components. This corresponds to the theoretical side of this work.

2. CONCLUSIONS

The following are the conclusions that were reached during this paper:
(a) The maximum likelihood estimator of variance components as follows:

$$\hat{\sigma}_e^2 = \frac{SS_e}{(1-u)(v+s)} = MS_E = 0.634 ,$$

$$\hat{\sigma}_\varphi^2 = \frac{SS_{\xi\times\eta}}{(v-1)(s-1)} - \hat{\sigma}_e^2 = MS_{\xi\times\eta} - MS_E = 29.087 ,$$

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$\hat{\sigma}_\theta^2 = \frac{SS_\theta - \hat{\sigma}_\varphi^2 - \hat{\sigma}_e^2}{s} = \frac{MS_\theta - MS_{\xi\times\eta} - MS_E}{s} = 85.471$ and
$\hat{\sigma}_\lambda^2 = \frac{(SS_\eta/s-1) - \hat{\sigma}_\varphi^2 - \hat{\sigma}_e^2}{v} = \frac{MS_\eta - MS_{\xi\times\eta} - MS_E}{v} = 45.184$.

a) The maximum likelihood estimators of variance components are unbiased as follows:
$E(\hat{\sigma}_e^2) - \sigma_e^2 = 0,\ E(\hat{\sigma}_\varphi^2) - \sigma_\varphi^2 = 0,\ E(\hat{\sigma}_\theta^2) - \sigma_\theta^2 = 0$ and $E(\hat{\sigma}_\lambda^2) - \sigma_\lambda^2 = 0$.

(b) The maximum likelihood estimators of variance components are biased as follows.

$$var(\hat{\sigma}_e^2) = \frac{\sigma_e^4}{(1-u)(v+s)},$$
$$var(\hat{\sigma}_\varphi^2) = \left[\frac{1}{(v-1)(s-1)} + \frac{1}{(1-u)(v+s)}\right]\sigma_e^4 ,$$
$$var(\hat{\sigma}_\theta^2) = \frac{1}{s^2}\left[\frac{(s\sigma_\theta^2+\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2)^2}{v(u-1)} + \left(\frac{1}{(v-1)(s-1)} + \frac{1}{(1-u)(v+s)}\right)\sigma_e^4\right] \text{ and}$$
$$var(\hat{\sigma}_\lambda^2) = \frac{v^2(p\sigma_\lambda^2+\sigma_\varphi^2+\sigma_\varepsilon^2)^2}{(s-1)} + \left(\frac{v^2}{(v-1)(s-1)} + \frac{v^2}{(1-u)(v+s)}\right)\sigma_e^4.$$

(c) The maximum penalized likelihood estimators of variance components as follows:

$$\hat{\sigma}_e^2 = \frac{SS_e}{(1-u)(v+s)+1} = 0.632 ,$$
$$\hat{\sigma}_\varphi^2 = \frac{SS_{\xi\times\eta} - \hat{\sigma}_e^2[(v-1)(s-1)+1]}{(v-1)(s-1)+1} = 14.499 ,$$
$$\hat{\sigma}_\theta^2 = \frac{sSS_\theta - (s(u-1)+1)(\hat{\sigma}_\varphi^2+\hat{\sigma}_e^2)}{s[(v-1)+1]} = 2009.471 \text{ and}$$
$$\hat{\sigma}_\lambda^2 = \frac{SS_\eta - [(s-1)+1] - (\hat{\sigma}_\varphi^2+\hat{\sigma}_e^2)}{v[(s-1)+1]} = 59.315 .$$

(d) The maximum penalized likelihood estimators of variance components for model (1) are biased.

$$E(\hat{\sigma}_e^2) - \sigma_e^2 = \frac{1}{(1-u)(v+s)+1}\big((1-u)(v+s)\big)\hat{\sigma}_e^2 - \hat{\sigma}_e^2 ,$$
$$E(\hat{\sigma}_\varphi^2) - \sigma_\varphi^2 = \frac{1}{(v-1)(s-1)+1}\Big[\big((v-1)(s-1)+1\big)(\hat{\sigma}_\varphi^2+\hat{\sigma}_e^2) - \big((v-1)(s-1)\big)\hat{\sigma}_e^2\Big] - \sigma_\varphi^2 ,$$
$$E(\hat{\sigma}_\theta^2) - \sigma_\theta^2 = \frac{s(\sigma_\varphi^2+\sigma_e^2) - (s(u-1)+1)\big(E[\hat{\sigma}_\varphi^2+\hat{\sigma}_e^2]\big)}{s[s(u-1)+1]} - \sigma_\theta^2 \text{ and}$$
$$E(\hat{\sigma}_\lambda^2) - \sigma_\lambda^2 = \frac{1}{v[v(s-1)+1]}E\left((s-1)(\sigma_\lambda^2) - E[\hat{\sigma}_\varphi^2+\hat{\sigma}_e^2]\right) - \sigma_\lambda^2 .$$

(e) The variance of maximum penalized likelihood estimators as follows:

$$var(\hat{\sigma}_e^2) = \frac{(v+s)(1-u)\sigma_e^4}{\big((1-u)(v+s)+1\big)^2},$$
$$var(\hat{\sigma}_\varphi^2) = \frac{(v-1)(s-1)\sigma_e^4 + [(v-1)(s-1)+1]v\frac{(v+s)(1-u)\sigma_e^4}{\big((1-u)(v+s)+1\big)^2}}{\big((v-1)(s-1)+1\big)^2},$$
$$var(\hat{\sigma}_\theta^2) = v\frac{1}{s^2}\left[\frac{(s\sigma_\theta^2+\sigma_\lambda^2+\sigma_\varphi^2+\sigma_e^2)^2 + (s(u-1)+1)^2(x+y)}{s[s(u-1)+1]^2}\right] \text{ and}$$

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$$var\left(\hat{\sigma}_{\lambda}^{2}\right)=\frac{\left(p\sigma_{\lambda}^{2}+\sigma_{\varphi}^{2}+\sigma_{\varepsilon}^{2}\right)^{2}+[(s-1)+1]^{2}(x+y)}{v[(s-1)+1]}.$$

where $x=var(\hat{\sigma}_{e}^{2})$ and $y=var\left(\hat{\sigma}_{\varphi}^{2}\right)$

(f) The mean square error of variance components of model (1) are as follows:

$MSE\left(\hat{\sigma}_{e,ML}^{2}\right)=0.00037\leq MSE\left(\hat{\sigma}_{e,MPL}^{2}\right)=0.001,$

$MSE\left(\hat{\sigma}_{\varphi,ML}^{2}\right)=0.328\leq MSE\left(\hat{\sigma}_{\varphi,MPL}^{2}\right)=49.998,$

$MSE\left(\hat{\sigma}_{\theta,ML}^{2}\right)=10471.091\leq MSE\left(\hat{\sigma}_{\theta,MPL}^{2}\right)=401452.322$ and

$MSE\left(\hat{\sigma}_{\lambda,ML}^{2}\right)=1782\leq MSE\left(\hat{\sigma}_{\lambda,MPL}^{2}\right)=342167.876.$

We see that the value of maximum likelihood estimators and man square error of variance components less than the mean square error and maximum penalized likelihood estimators of variance components. This corresponds to the theoretical side of this work.

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