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Maximizing Accuracy: Advancements in Numerical Methods for Ordinary Differential Equations

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Abstract: ***Euler's Method, Taylor's Method are the most fundamental and easiest methods to solve first order ordinary differential equations (ODEs). Many other methods like Runge-Kutta Method have been developed on the basis of these method. In this paper, the basic ideas behind Euler's Method, Taylor's Method, and Runge-Kutta Method, as well as the geometrical interpretation have been discussed. The main focus is confined to the mathematical interpretation and graphical representation of these method and to find a way to reduce the errors. In order to verify the accuracy of these methods, we compare numerical solutions to exact solutions. Numerical experiment and graphical representation of a specific problem have been discussed in this paper. MATLAB programs have been used for graphical representation and FORTRAN programs have been used for computational efficiency.***

Keywords: ***Euler's Method, Exact Solution, Numerical Solution, Runge-Kutta Method, Taylor's Method.***

1. INTRODUCTION

When it comes to modelling physical and natural processes with mathematics, differential equations are commonly used in the sciences and engineering. Differential equations are used to depict a wide variety of phenomena, including the motion of a simple harmonic oscillator, the deflection of a beam, and so on. These types of differential equations are quite complex and may not possess closed form solutions. In such cases they can be solved numerically. Numerical solutions to differential equations have become increasingly accessible in research, particularly with the rise of the computer. In the field of numerical methods, the objective is not to seek a correlation between the independent variable and the dependent variable, but rather to determine the numerical values of the dependent variable for specific values of the independent

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variable. There are several methods of describe various numerical methods for the solution of ordinary deferential equations such as: Euler's Method, Taylor's Series Method, Runge-Kutta Method, Modified Euler's Method, Picard's Method, Milne's Method. Now, among all of these methods of solution of ordinary Differential Equation, we are interested to discuss the three specific methods namely Euler's, Taylor's and Runge-Kutta Method.

2. METHODOLOGY

The closed form solutions to many ordinary differential equations encountered require us to look elsewhere for ways to develop solutions. One approach to tackle these tricky challenges is with the use of numerical techniques. In this chapter, we provide an overview of many numerical methods that may be used to solve numerous types of ordinary differential equations. We consider the problem of developing numerical methods to solve a first order initial value problem of the form,

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (1)$$

and then think about expanding these techniques to solve systems of ordinary differential equations of the type

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2 \ldots, y_m),\ y_1(x_0) = y_{10}$$
$$\frac{dy_2}{dx} = f_2(x, y_1, y_2 \ldots, y_m),\ y_2(x_0) = y_{20}$$
$$\frac{dy_m}{dx} = f_m(x, y_1, y_2 \ldots, y_m),\ y_m(x_0) = y_{m0} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (2)$$

Coupled systems of ordinary differential equations are sometimes written in the vector form

$$\frac{d\overrightarrow{y}}{dx} = \vec{f}(x, \vec{y}), \quad \vec{y}(x_0) = \overrightarrow{y_0}$$

Where $\vec{y}$, $\overrightarrow{y(x_0)}$ and $\vec{f}(x, \vec{y})$ are column vectors given.
We start with developing numerical methods for obtaining solutions to the first order initial value problem (1.) over an interval $x_0 \leq x \leq x_n$. Many of the techniques developed for this first order equation can, with modifications too also be applied to solve a first order system of differential equation.

(a) Euler's Method
Euler's method is a numerical technique to solve ordinary differential equations of the form

$$y' = f(x, y) \quad , \quad y(x_0) = y_0$$

So only first order ordinary differential equations can be solved by using Euler's method.
We consider the differential equation

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$$\frac{dy}{dx} = f(x, y) \ldots\ldots\ldots (1) \text{ with } y(x_0) = y_0 \ldots\ldots\ldots\ldots\ldots (2)$$

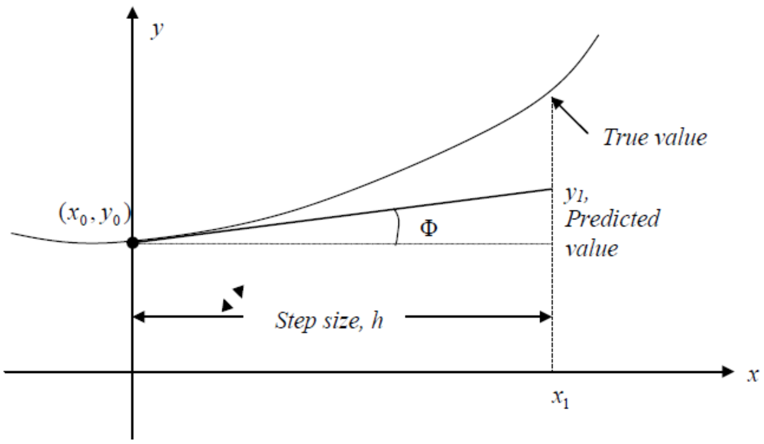

Figure: Graphical interpretation of the 1st step of Euler's method

Suppose that we want to solve the equation (1) for y at $x = x_r = x_0 + rh,\ \ r = 1, 2, 3, \cdots\cdots$ integrating (1) we obtain

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x, y)dx$$
$$\Rightarrow [y]_{y_0}^{y_1} = \int_{x_0}^{x_1} f(x, y)dx$$
$$\therefore y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Assuming that $f(x, y) = f(x_0, y_0\ )$ in $x_0 \leq x \leq x_1$,we get

$$y_1 = y_0 + (x_1 - x_0)f(x_0, y_0)$$
$$\therefore y_1 = y_0 + hf(x_0, y_0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Similarly for the range $x_1 \leq x \leq x_2$,we have $y_2 = y_1 + \int_{x_1}^{x_2} f(x, y)dx$

Assuming that $f(x, y) = f(x_1, y_1)$ in $x_1 \leq x \leq x_2$,we obtain $y_2 = y_1 + hf(x_1, y_1)$
Proceeding in this way, we get the general formula $y_{n+1} = y_n + hf(x_n, y_n),\ \ n = 0, 1, 2, 3, \cdots$
This method is called the Euler Method or Euler-Cauchy Method.

(b) Taylor's Series Method
The Taylor series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!} f''(a) + \cdots$$

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$$+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{(x-a)^n}{n!}f^{(n)}(a)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

In the most of the case, the Taylor's series converges to $f(x)$at every x and we often write the Taylor's series at $x=a$ as $f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots$

Instead of $f(x)$ and a, we prefer $y(x)$and x_0 and we may write $h=x-x_0$ in this case (1) becomes,

$y(x_0+h)=y(x_0)+y'(x_0)h+y''(x_0)\frac{h^2}{2!}+\cdots$

Hence, $y(x)=y(x_0)+(x-x_0)y'(x_0)+\frac{(x-x_0)^2}{2!}y''(x_0)+\cdots$

Derivation of Taylor's Series

We consider the first order differential equation $dy/dx=f(x,y)$ ············· (1)

with $y(x_0)=y_0$

Differentiating (1) with respect to x, we get

$d^2y/dx^2=\partial f/\partial x+\partial f/\partial y\cdot dy/dx$

$y''=f_x+f_y\cdot y'$ ············ (2)

Differentiating successively, we can get $y''',y^{iv}\cdots$

Putting the initial condition at $x=x_0$,then $y=y_0$, we get y_0',y_0'',y_0'''······

If $y(x)$ is the exact solution of (1) with the given initial condition, then, Taylor's series of $y(x)$ about $x=x_0$ is given by

$$y(x)=y(x_0)+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}y''(x_0)+\cdots$$

$$=y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}y_0''+\cdots\cdots\ (3)$$

Putting $x=x_0=x_0+h$ in (3), $y_1=y_0+hy_0'+\frac{h^2}{2!}y_0''+\cdots\cdots\cdots$ (4);

where $y(x_1)=y_1$

If the values of $y_0,y_0',y_0''\cdots$ are known, then (4) gives a power series for y_1.

Once y_1 is known ,we can compute $y_1',y_1'',\cdots$ from (1), (2) etc.

Then y can be expanded in Taylor's series about $x=x_1$, and

We have, $y(x_1+h)=y(x_2)=y_1+\frac{h}{1!}y_1'+\frac{h^2}{2!}y_1''+\frac{h^3}{3!}y_1'''+\cdots$

$$y_2=y_1+\frac{h}{1!}y_1'+\frac{h^2}{2!}y_1'+\frac{h^3}{3!}y_1'''+\cdots\cdots\ (5)$$

Proceeding in this way, we get, $y_{n+1}=y_n+\frac{h}{1!}y_n'+\frac{h^2}{2!}y_n''+\frac{h^3}{3!}y_n'''+\cdots\cdots$ (6)

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Where $y_n^r = (\frac{d^r y}{dx^r})_{(x_{n,}, y_n)}$

Here (6) is an infinite series and hence we have to truncate at some term to have the numerical value calculated.

(c) Runge-Kutta method
The most widely known member of the Runge–Kutta family is generally referred to as "RK4", "classical Runge–Kutta method" or simply as "the Runge–Kutta method".

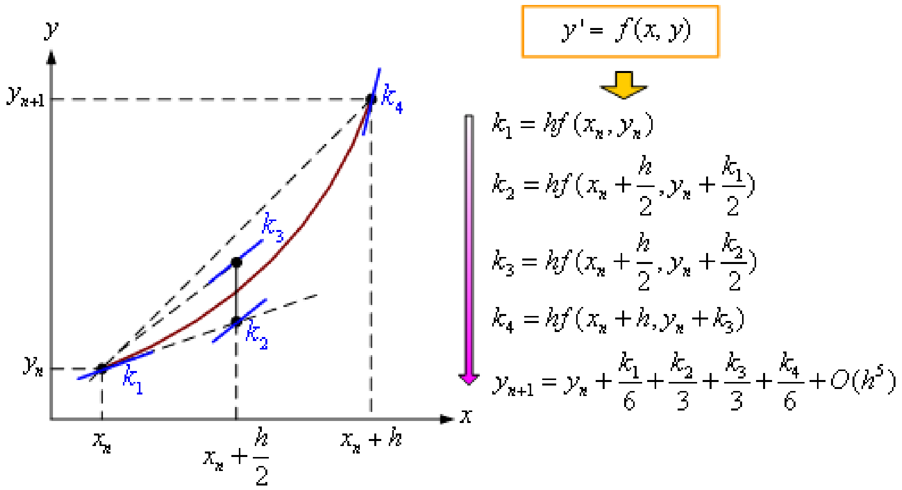

Figure: Graphical representation of RK-4

Let an IVP be specified as follows:
Consider $\frac{dy}{dx}$=$f(x, y)$ with inatial condition $y(x_0) = y_0$
Let h be the interval between equidistant values of x. By Taylor's series method, we have

$$y(x + h) = y(x) + hy'(x) + \frac{h^2}{2} y''(x) + O(h^3) \text{ ... ... (1)}$$

$\therefore y'(x) = f(x, y)$
y''=$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx}$= $f_x + f_{yy}$

Using these values in (1), we get
$y(x + h)$=$y(x)$+hf+$\frac{1}{2}h^2[f_x + f_{yy}]$+$O(h^3)$
$y(x + h) - y(x)$+hf+$\frac{1}{2}h^2[f_x + f_{yy}]$+$O(h^3)$
$\therefore \Delta y = hf$+$\frac{1}{2}h^2[f_x + f_{yy}]$+$0(h^3)$ ... ... (2)
Let $\Delta_1 y = k_1 = f(x, y)$; $\Delta x = h(x, y)$ ... ... (3)
$\Delta_2 y = k_2 = hf(x + ph, y + pk_1)$ ... ... (4)
And Δy=$p_1 k_1$+$p_2 k_2$ ... ... (5)

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Where p_1, p_2 and p are constant to be determined. From (4),
k_2=hf$(x + ph, y + pk_1)$
$=h[f(x,y)+(ph\frac{\partial}{\partial x}+pk_1\frac{\partial}{\partial y})f+\frac{\left(ph\frac{\partial}{\partial x}+pk_1\frac{\partial}{\partial y}\right)^2}{2!} \ldots\ldots\ldots\ldots\ldots\ldots\ldots]$
$=hf+phf_x+phff_y+\frac{\left(ph\frac{\partial}{\partial x}+pk_1\frac{\partial}{\partial y}\right)^2}{2!}$
$\therefore k_2=hf+ph^2(f_x+ff_y)+O(h^3) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$
Using (3) and(6)in (5),
we get $\Delta y=p_1hf+p_2[hf+ph^2(f_x+ff_y)+O(h^3) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$
From (2) and (7), we get
$p_1 + p_2=1, \quad pp_2=\frac{1}{2}$
Or, $p_1=1-p_2, \quad$ or $p=\frac{1}{2p_2}$

Using these values, (5) becomes $\Delta y= (1-p_2)k_1+p_2k_2$,where $k_1=hf(x,y)$
And $k_2=hf(x+\frac{1}{2p_2}h, y+\frac{hf}{2p_2})$
Now $\Delta y y(x+h)-y(x)$
Or $y(x+h)=y(x)+\Delta y$
Or $y(x+h) = y(x) + (1-p_2)k_1+p_2k_2$
$=y(x) + (1-p_2)hf++p_2hf(x+\frac{h}{2p_2}, y+\frac{hf}{2p_2})$
i.e $y_{n+1}=y_n+(1-p_2)hf(x_n,y_n)+p_2hf(x_n+\frac{h}{2p_2}, y_n+\frac{h}{2p_2}f(x_n,y_n))+O(h^3)$

which is the general **2nd order Runge-kutta method.**
The fourth order Runge –kutta formula is given by
$k_1 = hf(x,y)$
$k_2=hf(x+\frac{h}{2}, y+\frac{k_1}{2})$
$k_3=hf(x+\frac{h}{2}, y+\frac{k_2}{2})$
$k_4 = hf(x+h, y+k_3)$
And $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Which is known as **fourth order Runge-Kutta method**.
This fourth Runge –kutta method is mostly used in problems unless otherwise mentioned.
The second order Runge-Kutta formula:
Putting $p_1 = 0$ and $p = \frac{1}{2}$,
We get the following second order Runge-Kutta formula

$$k_1 = hf(x,y),\ k_2 = \left(x+\frac{h}{2}, y+\frac{k_1}{2}\right) \text{ and, } \Delta y = k_2 \text{ where } h = \Delta x.$$

The fourth order Runge-Kutta formula:

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$k_1 = hf(x,y)$, $k_2 = hf\left(x+\frac{h}{2}, y+\frac{k_1}{2}\right)$
$k_3 = hf\left(x+\frac{h}{2}, y+\frac{k_2}{2}\right)$, $k_4 = hf(x+h, y+k_3)$
and $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Numerical experiment:
In this section, we present the solution of an initial value problems using the Euler's method, Taylor's method and Runge-kutta method. To compare accuracy of the results we find the exact solution.
Numerical solution of the initial value problem $y' = y - x^2 + 1, y(0) = 0.5$ with $n = 10$ applying the Euler's method:

x_n	Approximate value of y_n	Exact values	Absolute value of error
0.0	0.5000	0.5000	0.0000
0.2	0.8000	0.8293	0.0293
0.4	1.1520	1.2140	0.0620
0.6	1.5504	1.6489	0.0985
0.8	1.9888	2.1272	0.1387
1.0	2.4582	2.6409	0.1827
1.2	2.9498	3.1799	0.2301
1.4	3.4518	3.7324	0.2806
1.6	3.9501	4.2834	0.3333
1.8	4.4282	4.8151	0.3870
2.0	4.8658	5.3055	0.4397

Numerical solution of the initial value problem $\frac{dy}{dx} = -y, 0 \leq x \leq 0.3, y(0)$with $h = 0.1$ applying the Taylor's method:

N	x_n	y_n	y_{exact}	errror
0	0.0	1.00	1.00	0
1	0.1	0.9	0.904837418	0.00483742
2	0.2	0.81	0.818730753	0.008730753
3	0.3	0.729	0.74081822	0.0111822

Numerical solution of the initial value problem $y' = y - x^2 + 1, y(0) = 0.5$ with $n = 10$ applying the RK-4 method:

x_n	Approximate value of y_n (Runge-Kutta order four)	Exact values	Absolute value of error
0.0	0.5000000	0.5000000	0

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0.2	0.8292933	0.8292986	0.0000053
0.4	1.2140762	1.2140877	0.0000114
0.6	1.6489220	1.6489406	0.0000186
0.8	2.1272027	2.1272295	0.0000269
1.0	2.6408227	2.6408591	0.0000364
1.2	3.1798942	3.1799415	0.0000474
1.4	3.7323401	3.7324000	0.0000599
1.6	4.2834095	4.2834838	0.0000743
1.8	4.8150857	4.8151763	0.0000906
2.0	5.3053630	5.3054720	0.0001089

Comparing and Graphical representation of Euler, Taylor & RK Methods with exact solution:

The differential equation $\frac{dy}{dx} = -y, 0 \leq x \leq 0.3, y(0) = 0$ with $h = 0.1$.
We use Euler, Taylor & RK Methods to compute $y(0.1), y(0.2), y(0.3)$ for the differential equation and compare the solution among them with exact solution.
Comparing table of the solution among Euler, Taylor & RK Methods of the above example is given below:

$\mathbf{n}$	$\mathbf{x_n}$	$\mathbf{y_n(exact)}$	$\mathbf{y_n(Euler)}$	$\mathbf{y_n(Taylor)}$	$\mathbf{y_n(RK2)}$	$\mathbf{y_n(RK4)}$
0	0.0	1.00	1.00	1.00	1.00	1.00
1	0.1	0.90483741	0.9	0.9	0.90484	0.90484
2	0.2	0.818730753	0.81	0.81	0.81873	0.81873
3	0.3	0.74081822	0.729	0.729	0.740817608	0.740817608

Graphical representation of the problem:

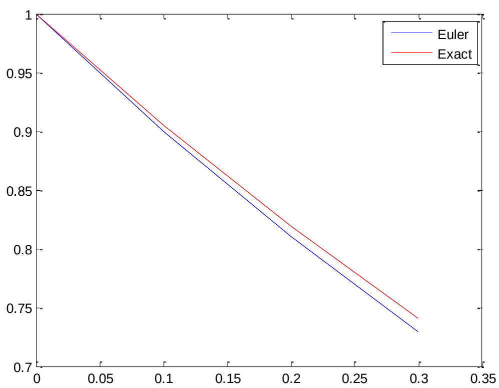

Figure: Comparison of Euler's Method and exact solution

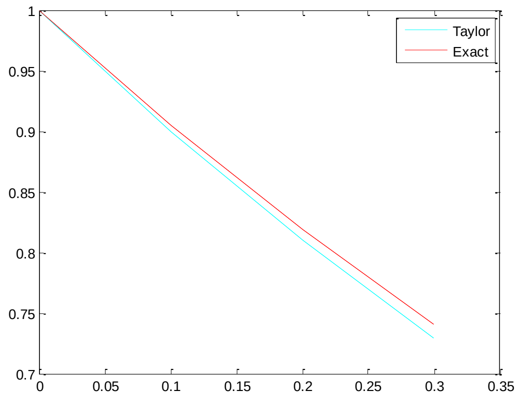

Figure: Comparison of Taylor's Method and exact solution

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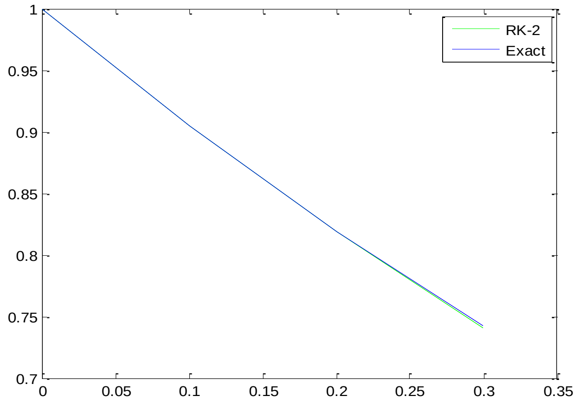

Figure: Comparison of RK-4 Method and exact solution

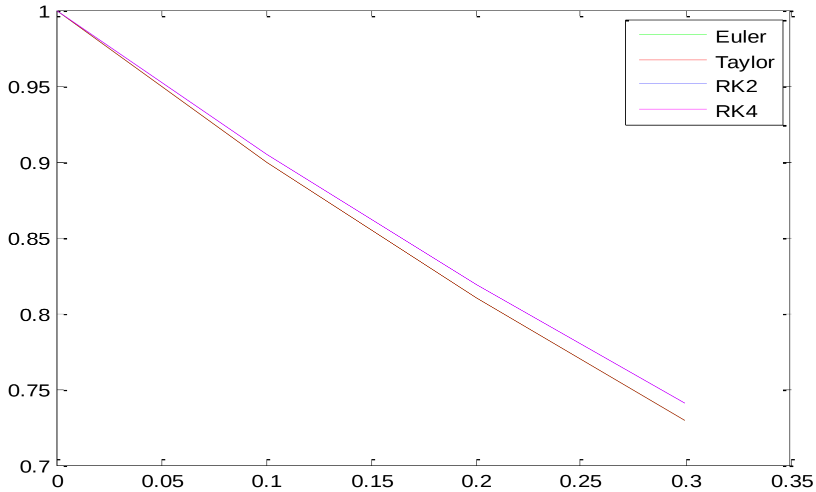

Figure: Comparison among Euler, Taylor & RK method

Comment:
From the above table and graphical representation, we can say that, the Runge-Kutta method gives the least error than any other of two methods. In this method, the error is too much least that, we are not able to identify the existence of graph of the error in the graphical representation. So, obviously we can say that, the Runge-Kutta method is the most appropriate and gives the most accurate result than any other method.

3. CONCLUSION

Differential equations are used in many different ways in the fields of science and engineering. So, it's important to figure out which numerical method is, all things considered, the best way to solve this differential equation. In this paper, there has tried to find out the most convenient method to solve ordinary differential equation. According to the discussion in three methods of numerical analysis named Euler's, Taylor's and Runge-kutta method, a decision has made that,

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the Runge-Kutta method is the most appropriate and the useful method for solving first order initial value problem of differential equation still now as per example. But the existence of this method to compute the solution of first order differential equation is not permanent. Because this result may be changed by any more accurate and useful method.

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