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Generalized Supplemented Semimodules

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Abstract: In this work generalized supplemented semimodules are defined which generalize generalized supplemented modules. We investigate some properties of these semimodules. We show that the finite sum of GS-semimodules is GS-semimodule. We also define WGS-semimodules and proved that a semiring S is semilocal if and only if every finitely generated semimodule is a WGS-semimodule. Furthermore, we prove that if A be a semimodule as well as $Rad(A) \ll A$. Then A is a WGS-semimodule if and only if $A/Rad(A)$ is semisimple.

Keywords: Generalized Supplemented Semimodule, Subtractive Semimodule, Lifting Semimodule.

1. INTRODUCTION

In 2006, Wang and Dang [11] defined generalized supplemented module. In this paper, we first introduce the concept of generalized supplemented semimodules. Firstly, let use point that, S shalt indicate an associtione semiring with neutrai and A shalt indicate an unitary left S-semimodule throughout this article. A (left) S-semimodule A is commutative additive semigroup with zero elements 0_A, jointly with a mapping from $S \times A$ into A (sending (s, a) to sa) such that $(r + s)a = ra + rs$, $r(a + b) = ra + rb$, $r(sa) = (rs)a$ and $0_a = s0_a = 0$ for all $a, b \in A$ and $r, s \in S$. Let N be sub set of A. We say that N is an S-subsemimodule of A, denoted by $N \leq A$, if and only if N is itself an S-semimodule with respect to the process for A [6]. A subsemimodule $N \leq A$ is called essential in A, denoted by $N \leq_e A$ (or $N \leq_e A$), if $N \cap L \neq 0$ for every non-zero subsemimodule $L \leq A$ [9]. A subsemimodule $N \leq A$ is said to be small or superfluous in A (writes $N \ll A$), if for every subsemimodule $K \leq A$ with $N + K = A$ assume that $K = A$ [10]. The radical of S-semimodule A, signified by $Rad(A)$, is the sum of all small subsemimodules of A [10]. A is said to be hollow, if all proper subsemimodules of A are small in A. A is called local if it contains the maximal single subsemimodule, the correct subsemimodule that contains all other subsemimodules. A is called simple if it has no nontrivial

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subsemimodules, and semisimple if A is a direct sum of its simple subsemimodules [8]. A semiring S is named semilocal semiring if $S/Rad(S)$ is semisimple. The socle of A meaning Soc(A), is the sum of all simple subsemimodules of A [8]. Let $U,K \leq A$. K is called a supplement of U in A if it is minimal with respect to $A = U + K$. A subsemimodule K of A is a supplement from U in A if and only if $A = U + K$ and $U \cap K \ll K$ [3]. A is supplemented if each subsemimodule U of A has a supplement in A. $U \leq A$ has ample supplements in A if each subsemimodules of A such that $A = U + K$ contains a supplement of U in A. A semimodule A is called amply supplemented if every subsemimodule from A has ample supplements in A. Hollow semimodules are amply supplemented [3]. $U \leq A$ is a subtractive subsemimodule of A if $a, a + b \in U$ then $b \in U$ [8]. If every $U \leq A$ is a subtractive, then A is named subtractive. If C is a subtractive subsemimodule, then A/C is an S-semimodule [6, p.165]. Section 2 is devoted to various properties of generalized supplement subsemimodules. We prove that if and only if A is a GAS-semimodule, then A is Artinian and confirms that DCC generalized to smaller subsemimodules. It is proved so as to every finite sum of GS-semimodule. In secation 3, we define WGS- semimodule. It is proved that S is semilocal iff each cyclic semimodule is a WGS-semimodule.

Lemma 1.1: Let A be a semimodule and V a supplement subsemimodule of A. Then $Rad(V) = V \cap Rad(A)$.

Proof: Assume V to be a supplement of $U \leq A$. Let $K \ll A$ and $Y \leq V$ with $(K \cap V) + Y = V$. The Then $A = U + V = U + (K \cap V) + Y = U + Y$, and so $Y = V$, i.e., $K \cap V \ll V$. This yields $V \cap Rad(A) \leq Rad(V)$, since $Rad(V) \leq V \cap Rad(A)$, we have $Rad(V) = V \cap Rad(A)$. □

In [6], [8] if A is a semimodule, then A represent Artinian if any non-empty set on subsemimodules of A contains minimal member in ration to setting inclusion. For present definition is equivalent to descending chain condition on subsemimodules of A.

Theorem 1.2: Take A a S-semimodule. If and only if A fulfills (DCC) on a small submodule, Rad(A) is then Artinian.
Proof: It is essentially the same as that of Theorem 5 in [2]. □

Proposition 1.3: [11, Proposition 14.22] (Semimodularity Law) Let A be an S-semimodule and let N_t and N_r be subsemimodules of A. Let F be a subtractive subsemimodule of A with $N_t \leq F$. Then $F \cap (N_t + N_r) = N_t + (F \cap N_r)$.

2. GS-semimodules and GAS-Semimodules

Definition 2.1: Let A be an S-semimodule and $F, B \leq A$. If $A = F + B$ and $F \cap B \leq Rad(B)$, then B is said to be a generalized supplement of F in A. If all subsemimodule of A has a generalized supplement in A, then A is named generalized supplemented semimodule or simply a GS-semimodule.

Definition 2.2: An S-semimodule A is said to be a generalized amply supplemented semimodule or simply a GAS-semimodule when $A = F + B$ (mean that F has generalized

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supplement $F' \leq B$). F is named a generalized supplement subsemimodule when F is a generalized supplement of some subsemimodule of A.

Remark 2.3: (1) Evidentially, all supplement is generalized supplement. Therefore, all supplemented semimodules are generalized supplemented. But the reverse include is incorrect. For instance, the Z-semimodule Q. Since $\mathrm{Rad}(Q) = Q$. While Q is not supplemented using example [5, 20.12].

(2) (Amply) supplemented semimodules and hollow semimodules are GS-semimodules. Let $S = N$ be a semiring of non-negative integers, and $A = N_8 = \frac{N}{8N}$. Then the semimodules S and A over a semiring S are local (hollow) semimodules and so are GS-semimodules.
The next can be thought of as a generalization of [4, Example 7.5]

Example 2.4: Let S be a Dedekind semidomain of the quotient semifield $K \neq S$. The S-semimodule $A = K^{(\Gamma)}$is generalized supplemented for all index Γ. If S is (a local Dedekind) semidomain, then A is supplemented only when Γ is finite. If S is a non-local Dedekind semidomain, as well A not supplemented for each index set Γ, because A not torsion, i.e., $\mathrm{Rad}(A) \neq A$.

Proposition 2.5: Take A be a GS-semimodule and N a subtractive subsemimodule of A such that $N \cap \mathrm{Rad}(A) = 0$. As well N is semisimple. Exclusively, a GS-semimodule A to $\mathrm{Rad}(A) = 0$ is semisimple.
Proof: Let $N^1 \leq N$. There exists $N^2 \leq A$ with $N^1 + N^2 = M$, $N^1 \cap N^2 \leq \mathrm{Rad}(N^2)$. So $N = N \cap A = N \cap (N^1 + N^2) = N^1 + (N \cap N^2)$, by Proposition 1.3. Since $N^1 \cap N^2 \leq \mathrm{Rad}\,(N^2)$ and $N^1 \cap N \cap N^2 = N^1 \cap N^2 \leq N \cap \mathrm{Rad}\,(N^2) \leq N \cap \mathrm{Rad}(A) = 0$, $N = N^1 \oplus (N \cap N^2)$. So that N would be is semisimple. □

Proposition 2.6: Take A subtractive GAS-semimodule also take Fdirect summand of A. As well F would be GAS-semimodule.
Proof: There exists $F' \leq A$ with $A = F \oplus F'$. Assume $F = B + H$, then $A = H + (B \oplus F')$. Since A is a GAS-semimodule, there exists $K \leq H$with $A = K + (C \oplus F')$ and $K \cap (B \oplus F') \leq \mathrm{Rad}(K)$. So $F = F \cap A = F \cap \left(K + (B \oplus F')\right) = K + B$, $K \cap B = K \cap (B \oplus F') \leq \mathrm{Rad}(K)$, This proves the claime. □

Proposition 2.7: Let A be a subtractive GS-semimodule. Then $A = F \oplus B$ to some semisimple F also some B with essential radical.
Proof: It is similar to the proof of Proposition 2.3 in [11]. □

Proposition 2.8: Let $A_1, U \leq A$ and A_1 be a GS-semimodule. If $A_1 + U$ contains a generalized supplement in A, furthermore U.
Proof: There exists $N_t \leq A$ with $N_t + (A_1 + U) = A$, $N_t \cap (A_1 + U) \leq \mathrm{Rad}\,(N_t)$. There exists $N_k \leq A_1$ with $(N_t + U) \cap A_1 + N_k = A_1$, $(N_t + U) \cap N_k \leq \mathrm{Rad}(N_k)$. So we have $N_t + U + N_r = A$ and $(N_t + U) \cap N_r \leq \mathrm{Rad}(N_r)$. It is sure that $(N_t + N_r) + U = A$. Since $N_r + U \leq$

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$A_1 + U, N_t \cap (N_r + U) \leq N_t \cap (A_1 + U) \leq Rad(N_t)$. Hence $(N_t + N_r) \cap U \leq N_t \cap (N_r + U)$ $+ \; N_r \cap (N_t + U) \leq Rad(N_t) + Rad(N_r) \leq Rad(N_t + N_r)$. So, $N_t + N_r$ be generalized supplement of U in A. □

Proposition 2.9: Let $A_1$and A_2 be GS-semimodules. If $A = A_1 + A_2$, then A is a GS-semimodule.
Proof: Let $U \leq A$. Since $A = A_1 + A_2 + U$ trivially contains a generalized supplement in A, $A_2 + U$ contains a generalized supplement in A using Proposition 2.8. Then also U using Proposition 2.8. □

Theorem 2.10: If A is a GS-semimodule, then $A/Rad(A)$ is semisimple.
Proof: Let $N_t \leq A$ with $Rad(A) \leq N_t$. Then $A = N_t + N_k$ and $N_t \cap N_k \leq Rad\ (A)$ for some $N_k \leq A$. So $A/Rad(A) = N_t/Rad(A) \oplus (N_k + Rad(A))/Rad(A)$, as well as all subsemimodule of $A/Rad(A)$ is a direct summand. □

Definition 2.11: A subsemimodule F of A is said to have generalized ample supplements in A if for all $H \leq A$ with $F + H = A$, F has a generalized supplement in H.

Proposition 2.12: If $A = A_1 + A_2$, and A_1, A_2 have generalized ample supplements in A, then $A_1 \cap A_2$ also has generalized ample supplements in A.

Theorem 2.13: The next are equivalent for a subtractive semimodule A with $B \leq A$.
(1) There is a decomposition $A = F \oplus F'$ with $F \leq B$ and $F' \cap B \leq Rad(F')$.
(2) There is a direct summand Fof A with $F \leq B$ and $B/F \leq Rad(A/F)$.
(3) B has a generalized supplement H in A with $H \cap B$ is a direct summand of B.

Proof: (1) $\Longrightarrow$ (2) Using the subtractiveness of F, we have A/F is a semimodule. $B/F \cong F' \cap B \leq Rad(F') \cong Rad(A/F)$. So $A = F \oplus F'$ and $B/F \leq Rad(A/F)$.
(2) $\Longrightarrow$ (1) If $A = F \oplus F'$ and $B/F \leq Rad(A/F)$, then $B = F + (F' \cap B)$, $F' \cap B \cong \frac{B}{F} \leq Rad\left(\frac{A}{F}\right) \leq A/F \cong F'$, consequently $F' \cap B \leq Rad(F')$.

(1) $\Longrightarrow$ (3) By hypothesis, F' the generalized supplement of B in A and $B = F \oplus (F' \cap B)$.
(3) $\Longrightarrow$ (1) Take H be a generalized supplement of B. Let $B = F \oplus (H \cap B)$. Then $A = B + H = F + (H \cap B) + H = F + H$and $F \cap H = (F \cap B) \cap H = F \cap (H \cap B) = 0$ (for $F \leq B$), i.e. F is a direct summand of A. □

Proposition 2.14: If each subsemimodule of A is a GS-semimodule, then A is a GAS-semimodule.
Proof: Let $F, B \leq A$ as well as $A = B + F$. There is $H \leq F$ with $F \cap B + H = F$, $(F \cap B) \cap H = B \cap H \leq Rad(H)$. So, $H + (F \cap B) = F \leq H + B$, so $B + F = A \leq H + B$. $A = H + B$. □

Corollary 2.15: The next are equivalent, for a semiring S.
(1) Each semimodule is a GAS-semimodule.

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(2) Each semimodule is a GS-semimodule.

Definition 2.16: [3] A semimodule A is named π-projective if for any two subsemimodules F and B of A with $A = F + B$, there is $\varphi, \gamma \in End(A)$ such that $\varphi + \gamma = 1_A$, $\varphi(A) \leq F$ and $\gamma(A) \leq B$.

Theorem 2.17: If A is a subtractive π-projective GS-semimodule, we get A is a GAS-semimodule.
Proof: Similar to the proof of theorem 3.16 in [3]. □

Theorem 2.18: A is Artinian if and only if A is a GAS-semimodule as well as fulfills (DCC) on generalized supplement subsemimodules with on small subsemimodules.
Proof: The first trend obviously. Reverse direction, assume A GAS satisfies (DCC). Thus $Rad(A)$ is Artinian by Theorem 1.2. Let $F \leq A$ and $Rad(A) \leq F$. There is a generalized supplement H of F in A, i.e., $A = F + H$, $F \cap H \leq Rad(H) \leq Rad(A)$. So $[\frac{A}{Rad(A)} = (F/Rad(A)) \oplus ((H + Rad(A))/Rad(A))]$. $A/Rad(A)$ is semisimple.

Now assume $Rad(A) \leq F_1 \leq F_2 \leq F_3 \leq \cdots$ represent ascending chain of subsemimodules of A. Since A is a GAS-semimodule, we can find a descending chain of subsemimodules $H_1 \geq H_2 \geq \cdots$ together with H_i is a generalized supplement of F_i in A to all $i \geq 1$. By assumption, there is positive integer r together with $H_r = H_{r+1} = H_{r+2} = \cdots$. Since $A/Rad(A) = F_i/Rad(A) \oplus (H_i + Rad(A))/Rad(A)$ for all $i \geq r$, it follows that $F_r = F_{r+1} = \cdots$. From now, $\frac{A}{Rad(A)}$ is Noetherian, so finitely generated. Therefore $A/Rad(A)$ is Artinian, □

Corollary 2.19: If A is finitely generated GAS-semimodule. Then A is Artinian iff A fulfills (DCC) on small subsemimodules.
Proof: "$\Longrightarrow$" is clear.
"$\Longleftarrow$" As $A/Rad(A)$ is semisimple and A is finitely generated, so $A/Rad(A)$ is Artinian. Since A fuifills (DCC) on small subsemimodules, $Rad(A)$ is Artinian using Theorem 1.2. Therefore, A is Artinian. □

Definition 2.20: [9] A semimodule A is called a lifting semimodule if for every subsemimodule $F \leq A$ of A there exist subsemimodules H, H' for A as well as $A = H \oplus H'$, $H \leq F$ and $F \cap H' \ll H'$.

Definition 2.21: [9] Let A be a subtractive semimodule. Then A is said to be a lifting semimodule, if for every subsemimodule $F \leq A$, there is a direct summand H of A and $H \leq A$ as well as $\frac{F}{H} \ll \frac{A}{H}$.

Theorem 2.22: If A be a subtractive semimodule as well as (ACC) on small subsemimodules. then, A is a GAS-semimodule as well as each generalized supplement is a direct summand of A if and only if A is lifting semimodule.

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Proof: ($\Longrightarrow$) Let $A = F + H$. There is $B \leq H$ with $A = F + B$ and $F \cap B \leq Rad(B)$. Since A together with (ACC) on small subtractive subsemimodules, $Rad(B)$ is Noetherian by [2]. So $Rad(B) \ll B$ by [7, Coro. 9.1.3], as well as B is a supplement of F. So A is amply supplemented. For any supplement is generalized supplement, any supplement is direct summand. A is lifting.

($\Longleftarrow$) Because A is lifting, A is an amply supplemented, from now A is a GAS-semimodule. Assume F be a generalized supplement, i.e., there is $H \leq A$ therefore $A = F + H$ while $F \cap H \leq Rad(F)$. With similar evidence from those of ($\Longrightarrow$), we are aware of that F is a supplement of H. Thus H is a direct summand of A, this proves the claime. □

Remark 2.23: Let A be a GS-semimodule and $Rad(A)$ be Noetherian. Then A is a supplemented semimodule.

3. WGS-semimodules

Definition 3.1: Let F, $H \leq A$. If $A = F + H$and $F \cap H \ll A$, then H is named a weak supplement of F in A. If all subsemimodule of A contains a weak supplement in A, then A is named a weakly supplemented semimodule.

Definition 3.2: A semimodule A is called generalized weakly supplemented or abbreviation writes a WGS-semimodule if for each subsemimodule $F \leq A$, there is $H \leq A$ together with $A = F + H$ and $F \cap H \leq Rad(A)$.

Proposition 3.3: Take A is a WGS-semimodule. Then.
(1) Each supplement subsemimodule of A is a WGS-semimodule.
(2) Each factor semimodule of A is a WGS-semimodule.

Proof: (1) Considered H be a supplement in A. To all $F \leq H$, because A is a WGS-semimodule, there is $B \leq A$ with $A = F + B$, $F \cap B \leq Rad(A)$. Therefore $H = H \cap A = H \cap (F + B) = F + (H \cap B)$ and $F \cap (H \cap B) = F \cap B = H \cap (F \cap B) \leq H \cap Rad(A) = Rad(H)$ using Lemma 1.1. Hence H is a GWS-semimodule.
(2) Take $B/F \leq A/F$. For $B \leq A$, there is $H \leq A$ together with $B + H = A$ and $H \cap B \leq Rad(A)$ since A is a WGS-semimodule. So, $A/F = B/F + (H + F)/$F. Let $\varphi: A \to A/F$ be a canonical epic. Since $H \cap B \leq Rad(A)$, $(B/F) \cap ((H + F)/F) = (B \cap (H + F))/F = (F + (H \cap B))/F = \varphi(B \cap H) \leq \varphi(Rad(A)) \leq Rad(A/F)$, A/F is a WGS-semimodule. □

Corollary 3.4: Take A be a semimodule and $F \ll A$. Then A is a WGS-semimodule if and only if $\frac{A}{F}$ is a WGS-semimodule.

Proposition 3.5: Take A is finitely generated. Then A is a WGS-semimodule if, and only if, A is weakly supplemented.
Proof: ($\Longleftarrow$) It's simple.

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($\Longrightarrow$) Suppose $N_t \leq A$, there exists $N_r \leq A$ with $N_t + N_r = A$ and $N_t \cap N_r \leq Rad(A)$ since A is a WGS-semimodule. Since A is finitely generated, $Rad(A) \ll A$ [10]. Hence $N_t \cap N_r \ll A$. □

Lemma 3.6: Let $H, A_1 \leq A$ and A_1 be a WGS-semimodule. If $A_1 + H$ has a generalized weak supplement in A, also H.

Proof: Suppose $F \leq A$ will be $(A_1 + H) + F = A$ and $F \cap (A_1 + H) \leq Rad(A)$. Since A_1 is a WGS-semimodule, there is a $B \leq A_1$ together with $A_1 \cap (F + H) + B = A_1$ and $B \cap (F + H) \leq Rad(A_1)$. Thus $A = H + F + B$ and $H \cap (F + B) \leq (H + A_1) \cap F + B \cap (F + H) \leq Rad(A)$, that is, $F + B$ is a generalized weak supplement of H in A. □

Proposition 3.7: Suppose $A = A_1 + A_2$. If A_1, A_2 are WGS-semimodules, then A is a WGS-semimodule.

Theorem 3.8: Let A be a semimodule as well as $Rad(A) \ll A$.Then the next are equivalent.
(1) A is a WGS-semimodule.
(2) $\frac{A}{Rad(A)}$ is semisimple.
(3) $A = A_1 \oplus A_2$ together with A_1 is semisimple, $Rad(A) \leq_e A_2$ and $A_2/Rad(A)$ is semisimple.

Proof: (1) $\Longrightarrow$ (2) Let $B \leq A$ with $Rad(A) \leq B$. Since A is a WGS-semimodule, there exists $F \leq A$ with $F + B = A$ and $F \cap B \leq Rad(A)$. From now $\frac{A}{Rad(A)} = B/Rad(A) + \frac{(F+Rad(A))}{Rad(A)}$ and $\frac{B}{Rad(A)} \cap \frac{(F+Rad(A))}{Rad(A)} = \big(B \cap F + Rad(A)\big)/Rad(A) = 0$.
(2) $\Longrightarrow$ (1) For any $F \leq A$, since $\frac{A}{Rad(A)}$ is semisimple, there is $B \leq A$ containing $Rad(A)$ as well as $\frac{A}{Rad(A)} = \frac{(F+Rad(A))}{Rad(A)} \oplus \frac{B}{Rad(A)}$. Hence $A = F + Rad(A\,) + B$. Since $Rad(A) \ll A$, $A = F + B$. $F \cap B \leq Rad(A)$ is clear.
(2) $\Leftrightarrow$ (3) By [11, Theorem 3.8]. □

Theorem 3.9: The next are equivalent, for a semiring S.
(1) S is semilocal.
(2) Each semimodule together with small radical is a WGS-semimodule.
(3) Each finitely generated semimodule is a WGS-semimodule.
(4) Each cyclic semimodule is a WGS-semimodule.

Proof: (1) $\Longrightarrow$ (2) Because every semimodule A there is a set (Γ) as well as an epimorphism $\varphi : S^{(\Gamma)} \longrightarrow A$ together with $\varphi(Rad(S^{(\Gamma)})) \leq Rad(A)$ and $\frac{S^{(\Gamma)}}{Rad(S^{(\Gamma)}) \cong (S/J(S))^{(\Gamma)}}$, an epimorphism exists $\xi : S^{(\Gamma)}/Rad(S^{(\Gamma)}) \longrightarrow \frac{A}{Rad(A)}$. $\frac{A}{Rad(A)}$ is hence semisimple. and therefore A is a WGS-semimodule by applying Theorem 3.8.
(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) the proof obviously.

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(4) ⟹ (1) By Proposition 3.5, since a semiring S is semilocal if and only if ${}_S S$ weakly supplemented. □

Example 3.10: Let p and q be prime numbers and consider the semiring $S = Z_{p,q} = \{\frac{x}{y} \mid x, y \in Z, y \neq 0, p \nmid y \text{ and } q \nmid y\}$, where S is a uniform semilocal Noetherian semidomain. Hence, ${}_S S$ is a WGS-semimodule by using Theorem 3.8. Because ${}_S S$ is Noetherian, it together with (ACC) to small subsemimodules. If ${}_S S$ a GS-semimodule, ${}_S S$ is therefore a supplemented semimodule using Remark 2.23, this results in a note of contradiction in [1, Example 2. 17].

4. REFERENCES

1. R. Alizade and E. Bugukasik, Cofinitely Weakly Supplemented Modules, Comm. Algebra, 31 (2003), 5377-5390.
2. I. Al-Khazzi, P. F. Smith, Modules with Chain Conditions on Superfluous Submodules, Comm. Algebra, 19 (1991), 2331-2351.
3. A. H. Alwan, Z. R. Sharif, ss-Supplemented semimodules, J. Interdiscip. Math. to appear.
4. E. Büyükasik, E. Mermut and S. Özdemir, Rad-supplemented modules, Rend. Semin. Mat. Univ. Padova, 124 (2010), 157-177.
5. J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, Lifting Modules. Supplements and Projectivity in Module Theory, Frontiers Math., (Birkhäuser, Boston, 2006)
6. J. S. Golan, semirings and their applications, Kluwer Acad. Publ., 1999
7. F. Kasch, Modules and Rings, Academic Press, London, 1982.
8. Y. Katsov, T. Nam, N. Tuyen, On subtractive semisimple semirings, Algebra Colloq. 16(3) (2009), 415-426.
9. Z. R. Sharif, A. H. Alwan, δ-Lifting and δ-Supplemented Semimodules, J. Optoelectron. Laser,41(8) 2022,164-171
10. N.X. Tuyen, H.X. Thang. On superfluous subsemimodules. Georgian Math J.; 10(4) (2003), 763-770.
11. Y. Wang and N. Ding, Generalized supplemented module, Taiwanese J. Math., 10 (2006), 1589-1601.
12. R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, Reading, 1991.